STABILITY STUDIES OF SIGMA PULSE FREQUENCY MODULATED CONTROL SYSTEMS

A thesis submitted

In Partial Fulfilment of the requirements

for the Degree of

MASTER OF TECHNOLOGY IN ELECTRICAL ENGINEERING

by

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to the

Department of Electrical Engineering

Indian Institute of Technology, Kanpur

December 1968

CERTIFICATE

Certified that this work on "Stability Studies of Sigma Pulse Frequency Modulated Control Systems" has been carried out under my supervision and this has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENT

The guidance received from Dr. M.A. Pai, under whose supervision this work was carried out is acknowledged most gratefully. The author expresses his sincere words of thankfulness to Dr. Pai for providing helpful suggestions, stimulating discussions and constant encouragement. He takes the opportunity of thanking Dr. Viswanathan for his most helpful suggestions in the design of the circuit. Thanks are due to Mr. N. Jagan Mohan Rao for his useful suggestions at the initial stage of the circuit-design. Thanks are also due to Mr. M.S. Varadarajan for time to time helpful discussions.

Last but not the least, the author wishes to thank all those who played no insignificant role in the completion of this work.

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ABSTRACT

Pulse frequency modulated (PFM) control systems have found their applications in such diverse areas as neuron modelling, incremental servos, attitude control of space-crafts etc. A comprehensive literature survey on various aspects of variable frequency sampling has been done. The main part of the thesis consists in the design and construction of a sigma pulse frequency modulator and using this modulator experimental studies have been done to examine stability properties of certain closed loop PFM control systems. It has been experimentally verified that sigma pulse frequency modulator exhibits the particular property of firing over a certain threshold. Apart from the experimental work, bounds for asymptotic stability of the origin in the large are established using Lyapunov's direct method. Furthermore, Popov's criterion has been applied to ensure the absolute stability. A number of examples have been worked out in detail to illustrate the method and the results are compared with experimental values.

CHAPTER - 1

PULSE FREQUENCY MODULATED CONTROL SYSTEMS

INTRODUCTION

In recent years a subslass of nonlinear sampled-data systems namely pulse frequency modulated (PFM) systems has attracted lot of attention by research workers. Besides the area of control systems, where its application has certain advantages over pulse width modulated systems, its importance is highlighted by the fact that the mathematical modelling of neuron networks results in a PFM system. In this chapter, the literature on this subject is reviewed and attention is focussed on some of the unsolved problems. In other chapters a special class of PFM systems namely sigma pulse-frequency control systems has been considered and is analysed from the view point of its stability properties.

This chapter is classified into the following sections (1) Generalised view of variable frequency sampling scheme (2) Motivation for study of PFM systems (3) Classifications of PFM systems (4) Methods of stability analysis and optimization of PFM control systems (5) Double schemes of modulation (6) Statistical approach and (7) Reconstruction of continuous signals.

1.1 Generalised View of Variable Frequency Sampling Scheme

The study of sampled-data system is based on considering sampling as a process of pulse modulation. The samples of the signal can be modulated in many different ways. Among these, the following four methods of sampling have been investigated in the literature. (1) Pulse amplitude modulation (PAM), where samples of the message function are taken at a rate exceeding twice the highest frequency present in the original signal and the output of the sampler is usually thought of as a sequence of equally spaced, equal duration pulses whose amplitudes are proportional to the input of the sampler at the sampling instants. The duration of the pulse is considered negligible compared to the smallest time constant of the system, hence the outputs may be approximated mathematically as impulses.(2) Pulse width modulation (PWM), where the output of the sampler is a train of equally spaced, constant amplitude pulses whose individual pulse duration varies as a function of the input to the sampler at the sampling instants. (3) Pulse position modulation (PPM), where positions of the pulses relative to regularly spaced reference instants are modulated by the message samples. (4) Pulse frequency modulation (PFM), where the output of the sampler is a train of asynchronous pulses. The instantaneous frequency of the pulses depends in some way on the input signal. This is essentially a variable frequency operation.

PAM is a linear process and systems having such sampling schemes have been studied extensively in the literature as linear

sampled-data (discrete time) systems. The sampling frequency which is fixed a-priori can be constant or periodically time varying. PWM and PPM are however inherently nonlinear processes. In control systems, PWM systems have also been studied although PPM scheme of modulation does not appear to have been studied. In all these schemes however the sampling pattern is fixed and is external to the system. In PFM, the instantaneous frequency depends on some function of a variable of the system, this is the point of departure from conventional sampled-data system. This is a nonlinear and variable frequency operation.

Primarily, all of the systems which had been investigated up to the year 1957 were those with a constant sampling frequency. In 1958, Hufnagel [1] introduced a procedure for the analysis of cyclic-rate (periodically time varying) sampled-data systems. This method was based on the modified Z-transform. The same system was discussed by Jury and Mullin [3] with the approach of difference equations, rather than the modified Z-transform as the starting point. Dorf, Farren and Phillips [4] considered sampling to be more efficient when similar output characteristic is obtained with fewer samples for a given observation interval and they discussed about a sampler whose sampling period is controlled by the absolute value of the first derivative of the error signal. In fact this approach of information processing is similar to the bandwidth saving techniques in television where the scan-speed is controlled so as to scan swiftly over areas of uniform tone and more slowly over areas of much tonal

discontinuity. The real importance of the variable frequency sampling was felt as a more efficient use of the computer in time-sharing the machine between several control tasks. This was discussed by Will [5]. The idea of increasing sampling efficiency using variable frequency was further discussed by Gupta [6,7] and quite recently by Ciscato and Mariani [11].

The literature in the PFM control systems till 1963 was scattered and did not consider the analysis of such systems from a generalised point of view. Since then, however, there has been a number of papers which have considered the problem in greater detail. The interest seems to be have been generated by the application of PFM systems in such diverse areas as neuron modelling, incremental servos, attitude stabilization etc. In the discussion that follows, the various aspects of PFM control systems are discussed in detail.

1.2 Motivation for Study of PFM Systems

It has been observed through experiments that when a stimulation is exerted upon the sensory organs, some form of energy absorption or transduction will occur, a signal is generated at the axon of the neurons and it is transmitted along nerve-fibre linking it to some reflex-centre where decision is made and control effort is issued in response to this stimulation. The neural signal in a single nerve-fibre is a train of asynchronous pulses of approximately identical shape. But its instantaneous frequency depends upon the

strength of the stimulus. So PFM signal is believed to exist in neural communication networks of physiological systems. There are hundreds and thousands of nerve-fibres in the body of animals and human-beings. Signals carried by these fibres are pulse frequency modulated. Hence the study of PFM in a single nerve-fibre constitutes a very drastic abstraction of the physiological case. It is for this reason that neuron-modelling, analysis of neural-network etc. have received great attention during past ten years by several persons namely Hodgkin and Huksley [12], Harmon [13], Goldstein [14], Jones [15], Hiltz [16], Jury and Pavlidis [17], Pavlidis [18], Caumon [19], Lybinskii and Poziw [20], Blanchard [21], Pavlidis [22], Jury and Blanchard [23], Pavlidis [24] etc. Further research in this domain of neuro-physiology can be oriented in two different directions, as a combined effort of bioengineers and physiologists. (1) Analysis of the neural-network is quite involved, mainly because of time varying threshold and lack of accurate modelling. So accurate models can be developed based on experimental observations. Investigation of neural properties as a function of different parameters such as temperature etc. will particularly facilitate in its more accurate analysis. (2) Optimal models can be defined with respect to certain performance criteria corresponding to certain behaviour of the neuron to be studied. Such a study will add to the knowledge of adaptive behaviour data handling and the process of transforming information.

Further applications of PFM lie in the uses of adaptive control [25], attitude control of space crafts and incremental servo systems [26,27,28].

1.3 Classifications of PFM Systems

We shall first define a generalised pulse frequency modulator (GPFM). A GPFM can be defined as a system which operates on piecewise continuous inputs and renders a sequence of asynchronous pulses having a certain shape but different signs depending on input signals. The time duration between two adjacent pulses is the carrier of information. In addition, the shape of the pulse also can carry information. Let us confine ourselves to the situation where the shape of the pulse is fixed a-priori. The instantaneous frequency of the pulse appearance depends in some way on the error signal. Depending upon the manner in which the frequency is computed by the system, we can have basically two different schemes of PFM.

1) PFM systems of the first type or discrete PFM (DPFM) - In this, the pulse train frequency is some function of the values of the error signal at discrete instants of time.

2) PFM systems of the second type (or sigma PFM (ΣPFM)) - Here, the pulse train frequency depends upon some functional of the error signal, defined on the interval between pulses.

1.4 Methods of Stability Analysis and Optimization of PFM Control Systems

To the author's knowledge, the earliest study of pulse frequency modulation was done by Ross [29] in 1949. In 1957,

Khurgin [30] considered a more general class of systems operating on random inputs and rendering a set of random impulses. The importance of this scheme of modulation was underlined by a number of researchers namely Harmon [13], Goldstein [14], Jones [15], Hiltz [16] and others in providing the model for neurons. Li [32], Meyer [33], Jones and Pinter [34] developed more accurate models of PFM schemes. Li considered a particular class of modulator known as integral pulse frequency modulator (IPFM) which had been defined as a device which emits an impulse whenever the absolute value of the integral of its input reaches a certain threshold level and then it resets the integral to zero. Li [32] and Meyer [33] introduced the concept of equilibrium in IPFM as one where pulses are emitted in fixed pattern, since in this case firing of pulses does not stop. They also discussed the property of "limit annulus" with inner and outer boundary of oscillations. Later on, this work was supported by a number of people working in this area [35,36,37,39,42]. In 1964, Pavlidis [38] considered a special class of modulation known as neuron pulse frequency modulator (NPFM) (also known as relaxation pulse frequency modulator RPFM) and proposed a quadratic type of Lyapunov function for investigating the stability of origin as the equilibrium point. He also discussed the stability of an equilibrium region.

Pavlidis and Jury [40] proposed a very general scheme of modulation referred as sigma pulse frequency modulation. Here, the error signal is fed to a higher order low-pass (nonlinear and time-varying) filter and the emission of a pulse occurs when the

output reaches a certain threshold level. The analysis of this class of system is very difficult and it is far from solved. However, the cases where the filters are of first order (linear and time-invariant) are amenable to analytical investigation. In fact this particular case of ΣPFM is the same as previously discussed NPFM. This special class of ΣPFM has improved stability and it can be easily realizable as compared to previously discussed IPFM. ΣPFM like few other nonlinear systems exhibits the phenomenon of 'limit annulus'. Pavlidis and Jury [40] developed a special quasi-describing function analysis for the study of sustained oscillations in ΣPFM. They also discussed about the irregular oscillations, showing the randomness of ΣPFM output. As a more generalisation to his earlier works, Pavlidis [45] introduced the concept of discontinuous dynamical systems. He extended Lyapunov's direct method for the stability investigation by choosing a positive definite function which is constant or decreasing along the trajectory during no pulse emission and decreasing when pulses are emitted. Quite recently, Pavlidis [46] gave a very systematic method for finding Liapunov function based on the differential equations containing impulses. It is important to note that frequency domain approach for absolute stability of ΣPFM using Popov's criterion has been considered quite recently by Dymkov [54]. Dymkov [55] also extended the harmonic balance method for studying sustained oscillations of IPFM by considering a sinusoidal input to the nonlinearity. This yet remains to be solved for a more general class of problem i.e. ΣPFM.

The schemes of PFM considered till now belong to the PFM schemes of second type. In 1966, Clark and Noges [43] examined a class of PFM system in which the time duration between two adjacent pulses is a function of the error at sampling instants. This is, in fact, a PFM system of the first type. They proved that PFM systems of the type considered can not be asymptotically stable at the origin. Hence the origin is an unstable point. However, system is stable outside of a small region enclosing the origin. So the concepts of Lagrange-stability were introduced. Beyond a certain sector, this class of PFM also exhibits the phenomenon of limit-annulus and irregular oscillations like ΣPFM. In fact this discrete PFM poses more stability problems rather than ΣPFM for which origin is globally stable within a certain sector. Kuntsevich and Chekhovoi [50] also discussed the stability of this discrete PFM. They considered a Liapunov function in the form of a positive definite Hermite quadratic form for ensuring a sufficient condition for absolute stability of the set of equilibrium positions of a system with a stable or neutral continuous linear portion and pulse frequency modulator with a refractory zone. As a more general approach, Chekhovoi [53] examined the nonlinear pulse systems with arbitrary single valued nonlinearities and arbitrary modulation law (PAM, PWM or PFM) and introduced the general notion of ultimate boundedness for such systems.

Recently a number of papers have appeared discussing more finer aspects of PFM control. Jury and Blanchard [47]

gave a nonlinear discrete equivalent for IPFM system and examined its stability by devising a step by step procedure for the construction of state trajectories. They showed this class of system exhibits instability, asymptotic stability in the large and Lagrange stability. The results were obtained for the plants having no zeros in their transfer functions, however, they can be extended to more general plants. Nonlinear discrete equivalent for ΣPFM has not yet been solved. Analysis of ΣPFM systems where the filters are of higher order, nonlinear and time-varying has not been attempted so far. Also a more accurate analysis in finding the inner and outer bounary of limit annulus exhibited by IPFM, ΣPFM and DPFM is needed.

Optimization with PFM System: In 1966, Dorf and Liang [56] considered a time optimal problem for the control of step motor with controlling pulses from the output of a pulse frequency modulator. Pavlidis [57] also dealt with this problem. Till this time, the problem did not attract much attention from many, since the maximum principle of Pontryagin was not applicable for this class of system. In 1967, Onysko and Noges [58] presented a modified Pontrayagin's maximum principle to determine the optimal polarity and positions of the pulses which make up the control function. However, this is applicable only for the open-loop systems. Further work is needed to apply optimization techniques to closed loop systems.

1.5 Double Schemes of Modulation

Controllers with PFM operate more effectively under conditions of small variations of the controlled variable,

while PWM operates more effectively under conditions of large variations. So with this in mind, Kuntsevich and Chekhovoi [60] considered a double scheme of modulation (PWM and PFM) and they investigated the stability of such systems on the basis of the discrete analog of a theorem of La-Salle. The method may also be used for the special cases as PFM alone and PWM alone. Because of the generality and inherent advantages, this class of modulation has become a topic of recent interest. It should be noted that under this double scheme of modulation, we can have three possible schemes (1) PWM and IPFM (2) PWM and ΣPFM and (3) PWM and DPFM. Kuntsevich and Chekhovoi considered the third type of scheme. Other two combinations are also of equal interest. They can be taken as future research problems. The optimization techniques for all these systems may also find their application in future. In fact the idea of double scheme of modulation with PAM and PWM (also known as delta modulation [59]) was thought a few years back. The optimization for this double modulation (PAM and PWM) has quite recently been investigated [61].

1.6 Statistical Approach

If we consider an IPFM with deterministic input and fixed threshold, the output is a known sequence of asynchronous pulses. If we consider a general PFM system with higher order, nonlinear, time varying filter, the output of the modulator is a sequence of random pulses even with the known inputs. So in these cases, one has to worry about the knowledge of the

statistics of (t_n), the random point process whose variates represent the sampling time. The occurance of a random pulse-train has been observed in various other situations. In the study of vacuum tubes, one encounters the shot-noise process defined as $y(t) = \sum_{-\infty}^{+\infty} h(t - t_n)$ where the firing times t_n constitute a poisson point process. Similarly Barkenhouse noise which occurs during the magnetisation of a ferromagnetic material can also be described by a random train of pulses. So to include such systems, it is quite natural to study a general class of problem where we have a system which operates on random inputs and yields randomly spaced pulses. This type of problem is really difficult and tedious to study. Leneman [64,65,66,67,68] considered a random process consisting of infinite number of pulses occuring at random times with random intensities. He evaluated the first order and second order statistical properties of impulse processes. Such problems were also studied by Gupta and Jury [63], Lee [62], Beutler [67]etc.

In general, PFM presents a high degree of noise-immunity. But we do not know the degree of noise-immunity in a very quantitative sense. In 1967, Hutchinson and Chon [71] found the statistical properties of the output y(t) of the modulator when the input x(t) to the modulator is a Gaussian White noise process. They also discussed the effect of parameters of PFM systems on the noise. Problems for extending these results for

different types of noise e.g. non-White Gaussian, non-Gaussion, signal plus noise input etc. can be taken as future research problems.

It should be noted that output of the PFM systems indicates random-oscillatory-pattern. So a certain relation can be found between a random process and the output of a PFM system. This problem was indicated by Pavlidis and Jury in their paper [40]. The relation between random process and output of some nonlinear discrete systems was, earlier, discussed in detail by R.E. Kalman [76].

1.7 Reconstruction of Continuous Signals

One of the most important problems in sampled-data is to reconstruct a continuous signal from its samples. The optimum recovery of a continuous signal from unequally spaced samples appearing at the output of a pulse frequency modulator is an important problem. Recently papers have appeared on the problem of reconstructing a continuous signal from the samples taken at periodically non-uniformly spaced intervals[72,73,74,75]. The optimum mean square error time varying reconstruction filter in the presence of jitter for the periodically varying sampling rate has been determined by B.Liu and Franaszek [74].

1.8 Conclusions

In this chapter a survey of literature on almost all the aspects of pulse-frequency modulation has been attempted. A number

of research problems are still open in this area. With this, it is believed that a new burst of research activities in the challenging field of PFM control system will be materialised.

CHAPTER - 2

A CIRCUIT DESIGN FOR A SIGMA PULSE FREQUENCY MODULATOR

INTRODUCTION

In the previous chapter a general pulse-frequency-modulator has been defined as a nonlinear operator which operates on piece-wise continuous inputs and renders a train of pulses with the following properties (1) The shape of the pulses is fixed a-priori. In many applications when the shape is of no importance, $P_o(t)$ a function of time describing the pulse-shape can be regarded as an impulse function (2) The pulses are numbered by an integer n ($n > 1$). The nth pulse is completely characterised by its emission time t_n and sign $\epsilon_n (\epsilon_n = \pm 1)$.

Fig.1(a) shows a block diagram of general PFM, Fig.1(b) and Fig.1(c) show the input and the output patterns of the block. The time duration between two adjacent pulses T_n depends on the error signal either at the sampling instant t_n or during the interval between two pulses. When it depends upon the magnitude of the error at discrete instant t_n, we have the scheme as PFM of the first type or discrete PFM. We consider PFM of the second type where time between adjacent pulses depends upon the error signal during the interval between these adjacent pulses. This is called sigma PFM.

If $T_n = t_{n+1} - t_n$

1) $T_n = f(x_{t_n})$ ——— PFM of the first type (discrete PFM)

2) $T_n = g(x(t)) \quad t_n \leq t \leq t_{n+1}$ —— PFM of the second type (Sigma PFM).

Discussion is confined to sigma PFM modulators in this chapter. First, the equivalent circuit of ΣPFM is explained then, a design of ΣPFM with its extensions and limitations is discussed.

2.1 Equivalent Circuit of ΣPFM [41,45,54]

Fig.2 explains the equivalent circuit of ΣPFM.

Here, $x(t)$ is the input (usually error signal) to the modulator,

a is the feedback constant,

r is the threshold level (which is constant),

$p(t)$ is the output of the integrator,

$m(t)$ is the output of the modulator.

Integrator (transfer function $= \frac{1}{s}$) with a in the negative feedback loop around it is equivalent to a first order low pass filter. The transfer function of this filter $= \frac{1}{s + a}$. So the modulation process can be described as simply passing the input through a low pass filter and then sensing when the output of the filter $p(t)$ reaches the threshold $\pm$ r. When $p(t)$ reaches the threshold the output $m(t)$ is an impulse function and the integrator is immediately reset to zero and the process begins again. The sign of the output pulse is determined by the sign of $p(t)$. Mathematically this process can be described by a set of equations

$$\dot{p}(t) = -ap(t) + x(t) - r\delta(|p| - r)\,\mathrm{sgn}(p). \qquad (2.1.1)$$

$$m(t) = \mathrm{Sgn}(p)\ \delta\,(|p| - r) \qquad (2.1.2)$$

It should be noted that there are two parameters of this modulation process.

1) Threshold constant, r

2) Feedback constant, a

A very special case is when $a = o$, this defines Integral Pulse Frequency Modulator (IPFM).

2.2 Circuit Design of ΣPFM

Consider the circuit shown in Fig.3.

The transfer function of the R-C circuit $= \dfrac{\frac{1}{Cs}}{R + \frac{1}{Cs}}$

i.e. filter transfer function $= \dfrac{1}{RCs + 1} = \dfrac{a}{s + a}$

(where $\dfrac{1}{RC} = a$)

a, in the numerator can be regarded as a linear gain of the process of modulation.

Now consider an error signal x(t). See Fig.3(a). p(t) is the filter output which is the input to the unijunction transistor (UJT). UJT breaks at a certain positive potential V_p (threshold level r). V_p depends upon the UJT supply B_+ and resistances R_1 and R_2 and interbase resistance. So for a fixed setting of R_1, R_2, B_+ and a particular transistor, the peak voltage V_p or threshold r is a fixed quantity. Hence, whenever p(t) reaches the level r, capacitor C discharges through R_1, a triggering signal at B_1 can be observed. Discharging time constant can be

chosen very small compared to the charging time constant. This process repeats again. So the output m(t) will be a sequence of triggers as shown in Fig.3(c). Fig.3(b) is equivalent to the circuit shown in the Fig.4. This circuit achieves ΣPFM type of modulation for an error signal which is always positive.

It should be noted at this point that output at the base B_1 requires shaping by a suitably designed monostable, if these pulses have to control some physical plant.

Now, consider a signal x(t) which varies all the way from positive to negative. See Fig.1(b). Since, UJT fires only for positive threshold, we have to go for two such channels so as to get a sequence of positive and negative pulses in response to alternating error inputs. Whenever a pulse occurs in any of the channels, both the capacitors are reset to zero. See Fig.5. This is essential in order to achieve the required scheme of modulation. It should be noted that both the channels should break at the same threshold. Pulse output from both the shaping elements should have the same area. The pulse area will simply contribute to the linear gain of the plant. Fig.5 is equivalent to Fig.6 and Fig.7. a and r as defined earlier are the parameters of the process of modulation.

A designed circuit has been shown in Fig.8. Following points may be noted.

Instead of picking the positive trigger at B_1 and directly iving it to monostable for shaping, we pick up the trigger at ase B_2 and pass it through a capacitor and then invert it for

driving the PNP monostable. This is done so to improve the discharge of capacitor C by directly grounding B_1 of UJT.

2) Inverters no.2 and 3 can be eliminated by choosing NPN monostables and changing the discharge circuit accordingly.

3) The height of the pulse depends upon the collector supply voltage and output impedance of the monostable. The duration of the pulse depends upon the values of R_o and C_o. It can be shown that width of the pulse is approximately equal to $0.69\ R_oC_o$. Hence by a suitable adjustment of R_o, C_o, collector supply and R_c, we can control the area of the pulse.

4) The threshold r can be varied by changing B_+ and the resistance in base B_2. Mathematically, $r = \eta V_{BB} + V_D$ where V_{BB} is the interbase voltage which will depend upon B_+ and resistances in UJT bases; η is the intrinsic standoff ratio; V_D is equivalent emitter diode voltage which is quite negligible. So $r \simeq \eta V_{BB}$, η varies from 0.6 to 0.64 on an average.

5) Parameter a can be decided by the choice of R and C. We have the relation $a = \frac{1}{RC}$.

2.3 Circuit Response to a Constant Stimulus

Consider Fig.2. The filter transfer function is given by $\frac{1}{s+a}$. If we use a constant stimulus x_o(i.e. $x(t) = x_o$), we obtain the following equation for determining t_o, the firing interval,

$$\int_0^{t_o} e^{-at}\, x_o\, dt = r \qquad (2.3.1)$$

(for IPFM, substituting $a = 0$, $\int_0^{t_o} x_o\, dt = r$)

Substituting $x(t) = x_0$ = constant, we have,

$$x_0 \int_0^{t_0} e^{-at}\, dt = r$$

or $$e^{-at_0} = (1 - \frac{ar}{x_0})$$

or $$t_0 = -\frac{1}{a} \mathrm{Log}_e(1 - \frac{ar}{x_0}) \qquad (2.3.2)$$

It can be immediately noticed from equation (2.3.2) that modulator will never fire if $x_0 \leq ar$. This (ar) is also known as the input threshold or the rheobase of the modulator.
If we consider the filter with gain a i.e. $\frac{a}{s+a}$, we shall get by a similar process

$$t_0 = -\frac{1}{a} \mathrm{Log}_e(1 - \frac{r}{x_0}) \qquad (2.3.3)$$

So condition for firing would be $x_0 > r$.
For a particular setting of 'a' and 'r', we can get the plot t_0 vs. x_0. This plot is known as strength duration curve, Fig.9(a) explains the method for determining the firing time t_0 for a given stimulus and threshold. Fig.9(b) is a plot of strength duration curve (t_0 vs. x_0). For the stimulus above the threshold, firing occurs and for the stimulus below or equal to the threshold, firing does not occur. This particular property is one of the most important properties of sensory receptors. ΣPFM with constant threshold exhibits this property. In fact, other properties such as gradient threshold and adaptation are exhibited by ΣPFM with time varying thresholds. This has been discussed in detail by Jury [23].

Experimental observations for determining strength duration curve:

Practical values chosen, R = 100 K, c = 10 μf,

Pulse height = 10 volts, Pulse duration = 0.1 sec.

Threshold r = 6 volts.

Table 1

x_o in volts	experimentally observed firing time t_o in seconds	theoretical t_o in seconds. $t_o = -\frac{1}{a}\text{Log}_e(1-\frac{r}{x_o})$
8	1.2	1.38
10	0.8	0.92
12	0.5	0.693
14	0.4	0.615
18	0.26	0.4
22	0.2	0.314
30	0.15	0.223
40	0.1	0.148

At a particular stimulus x_o equal to 8 volts, records of x(t), m(t), $p_1(t)$ and $p_2(t)$ were taken. See Fig.10. The graph of stimulus strength versus firing time i.e. strength duration curve was drawn. See Fig.11.

2.4 Circuit Response to a Sinusoidal Input

A low frequency sinusoidal signal was fed into the modulator. The response is dependent upon the threshold and the level of input amplitude. A record was taken for x(t), m(t),

$p_1(t)$, $p_2(t)$ with the following specifications. See Fig. 12.

Pulse height = 10 volts, Pulse duration = 0.1 sec.

Threshold = 6 volts, Amplitude of sinusoidal error = 12 volts,

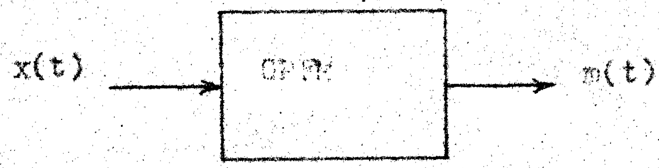
Frequency of error input = 1 radian/sec.

The response m(t) shows two pulses per half period. We get a sequence of positive and negative pulses in response to an alternating input. It may be noted that lowering the threshold and increasing the amplitude level increases the number of pulses per half period. The circuit response is dependent upon the frequency of the input signals.

2.5 Conclusions

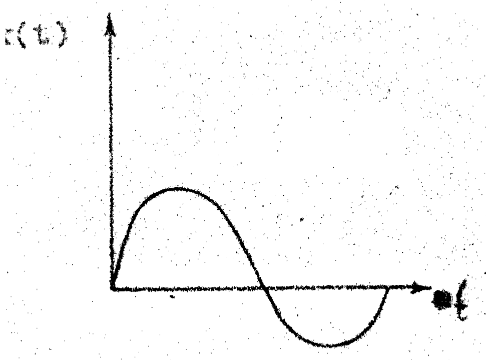
The circuit as given represents a general circuit for sigma pulse frequency modulator from the following considerations. 1) This responds to all types of our error signals, positive, negative and alternating. However, this is responsive only to low frequency signals. 2) The same circuit can be extended to Integral Pulse Frequency Modulator where RC is quite large such that the filter transfer function is $\frac{1}{RCs}$. Increasing the RC arbitrarily large results in marked attenuation of the signal. However, this can be overcome by putting an amplifier with a gain of RC before the filter.

In the next two chapters stability of pulse frequency modulated control system using this scheme of PFM is discussed experimentally as well as theoretically.



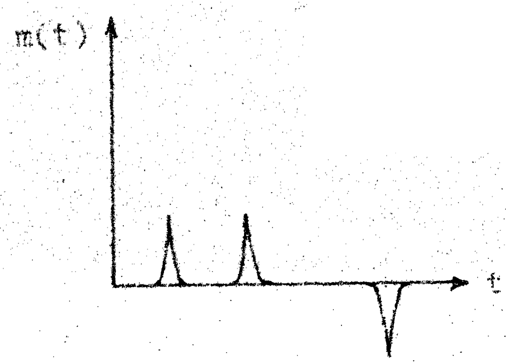
Block Diagram of a General Pluse Frequency modulator

FIG.1(a)



Input to the modulator

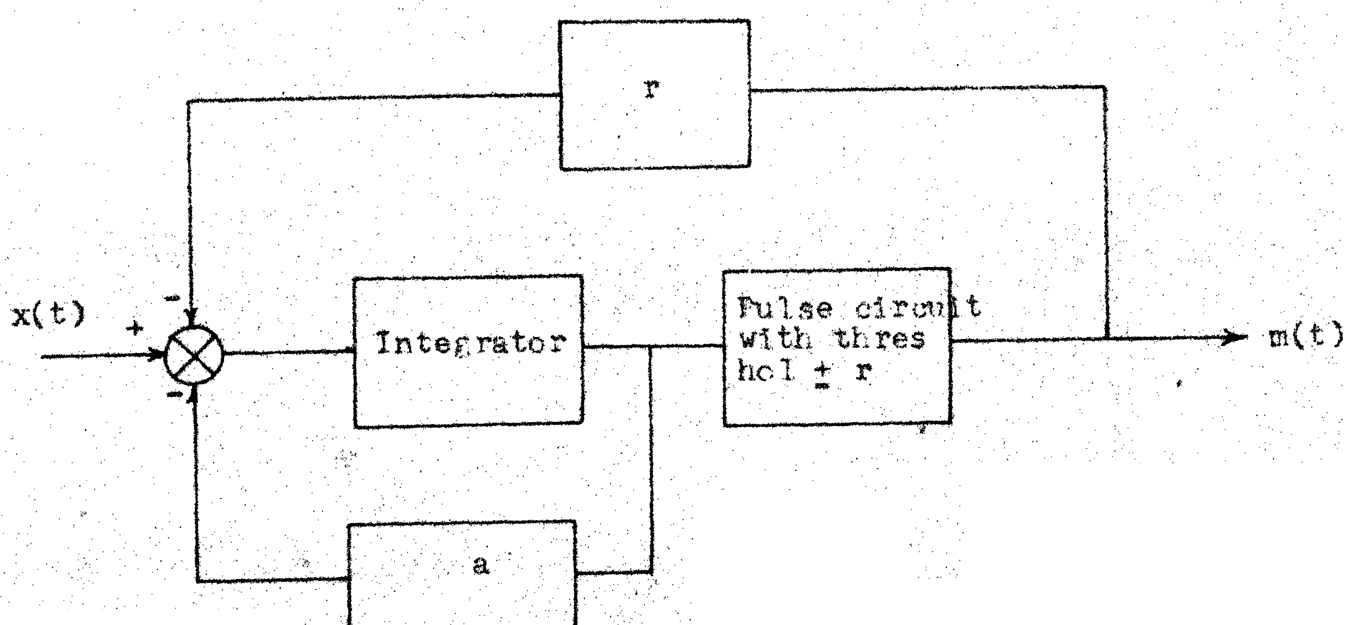
FIG.1(b)



Output of the modulator

FIG.1(c)

Fig. 1



Equivalent circuit of CPFM

FIG.2

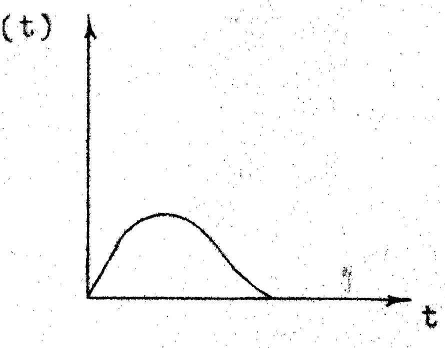


Fig.3(a)

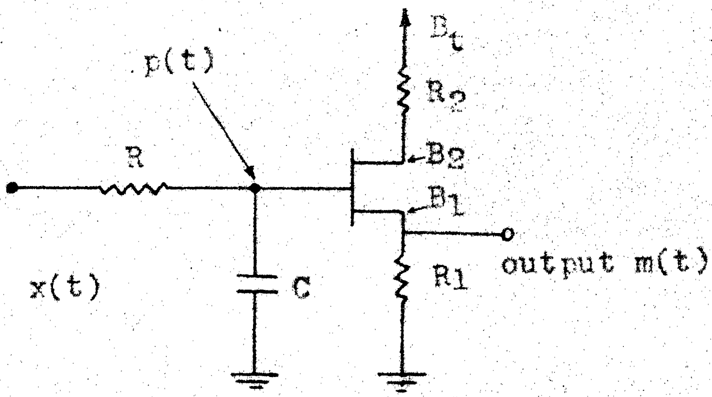


Fig.3(b)

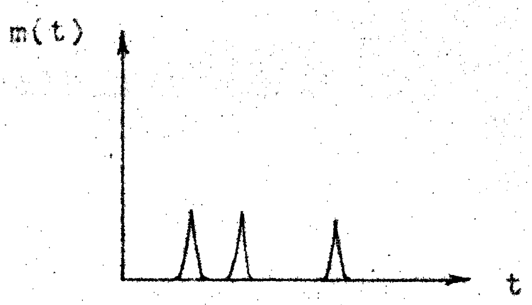
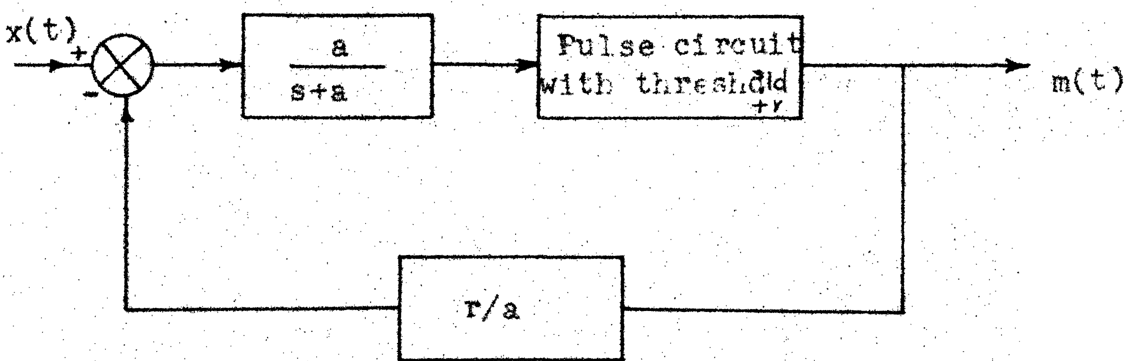


Fig.3(c)

FIG.3



Equivalent circuit of fig.3(b)

FIG.4

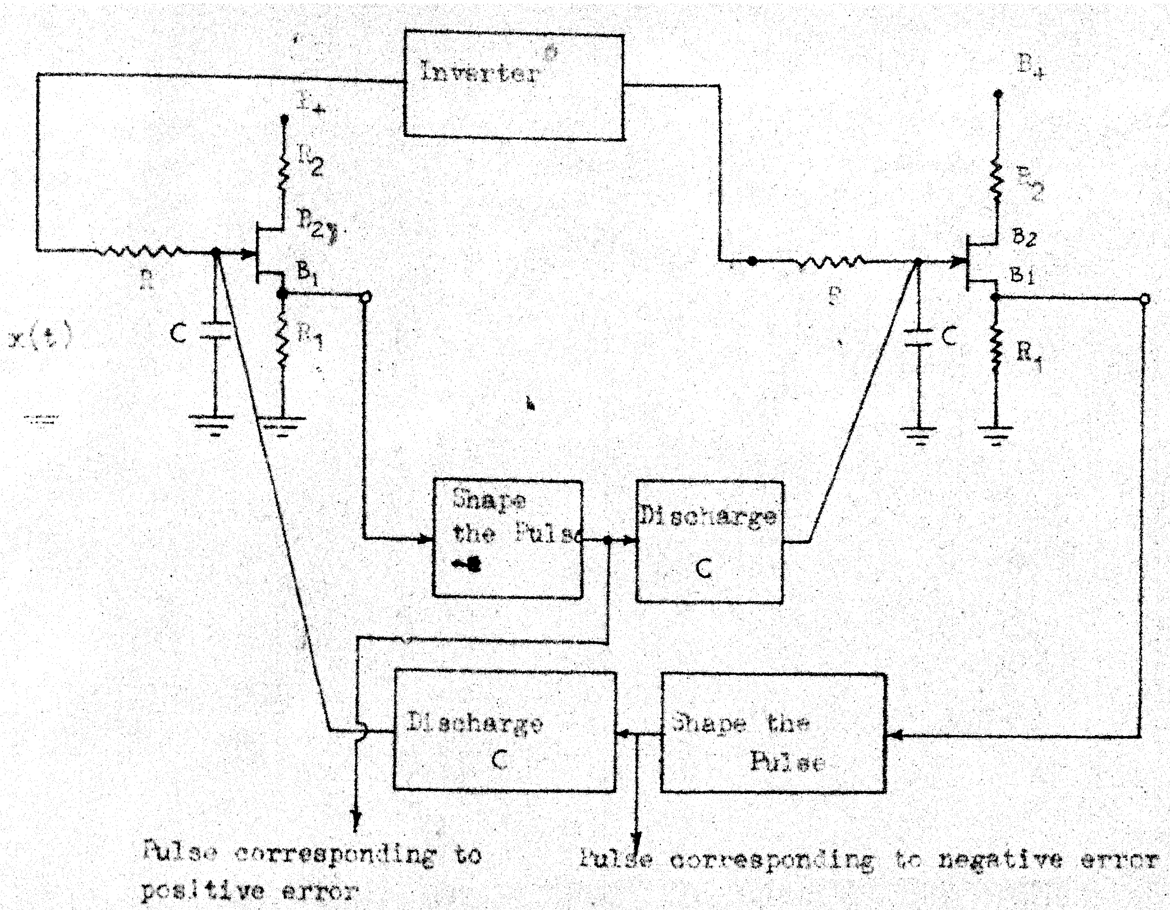


FIG. 5 Sigma Pulse Frequency modulator Cicuit

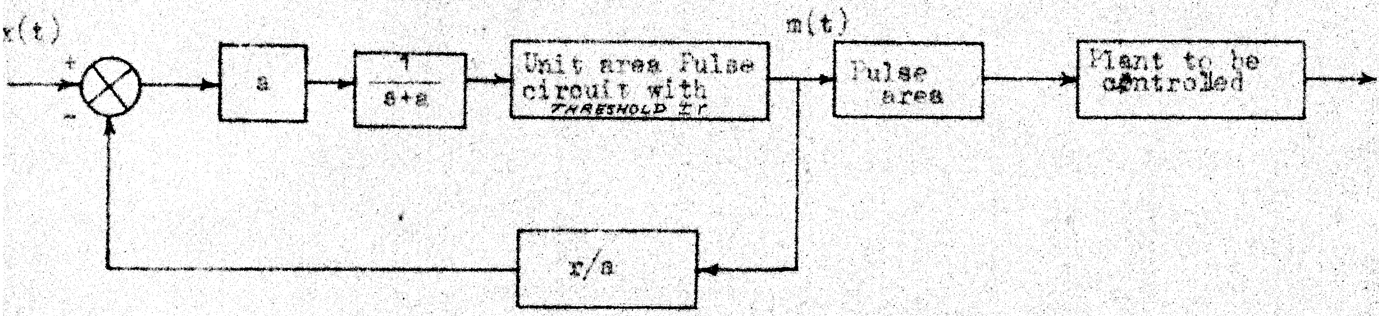


FIG. 6 Equivalent circuit of Fig.5

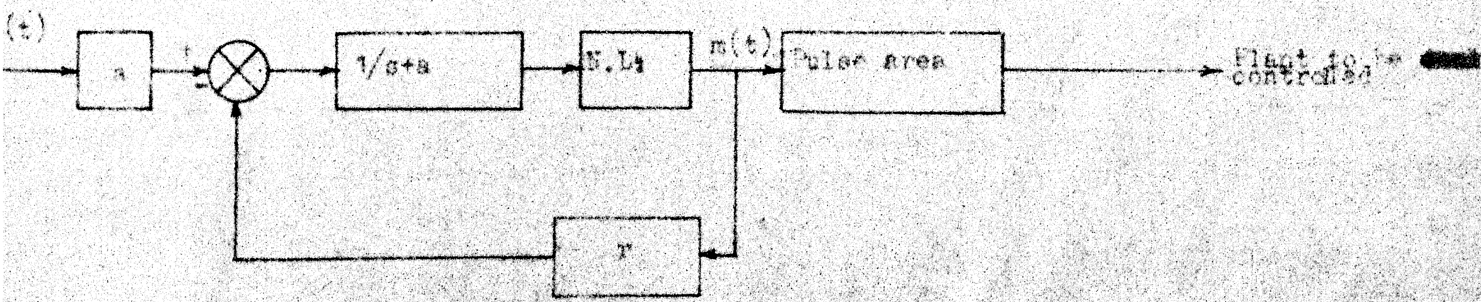


FIG.7 Simplified equivalent circuit of Fig. 5

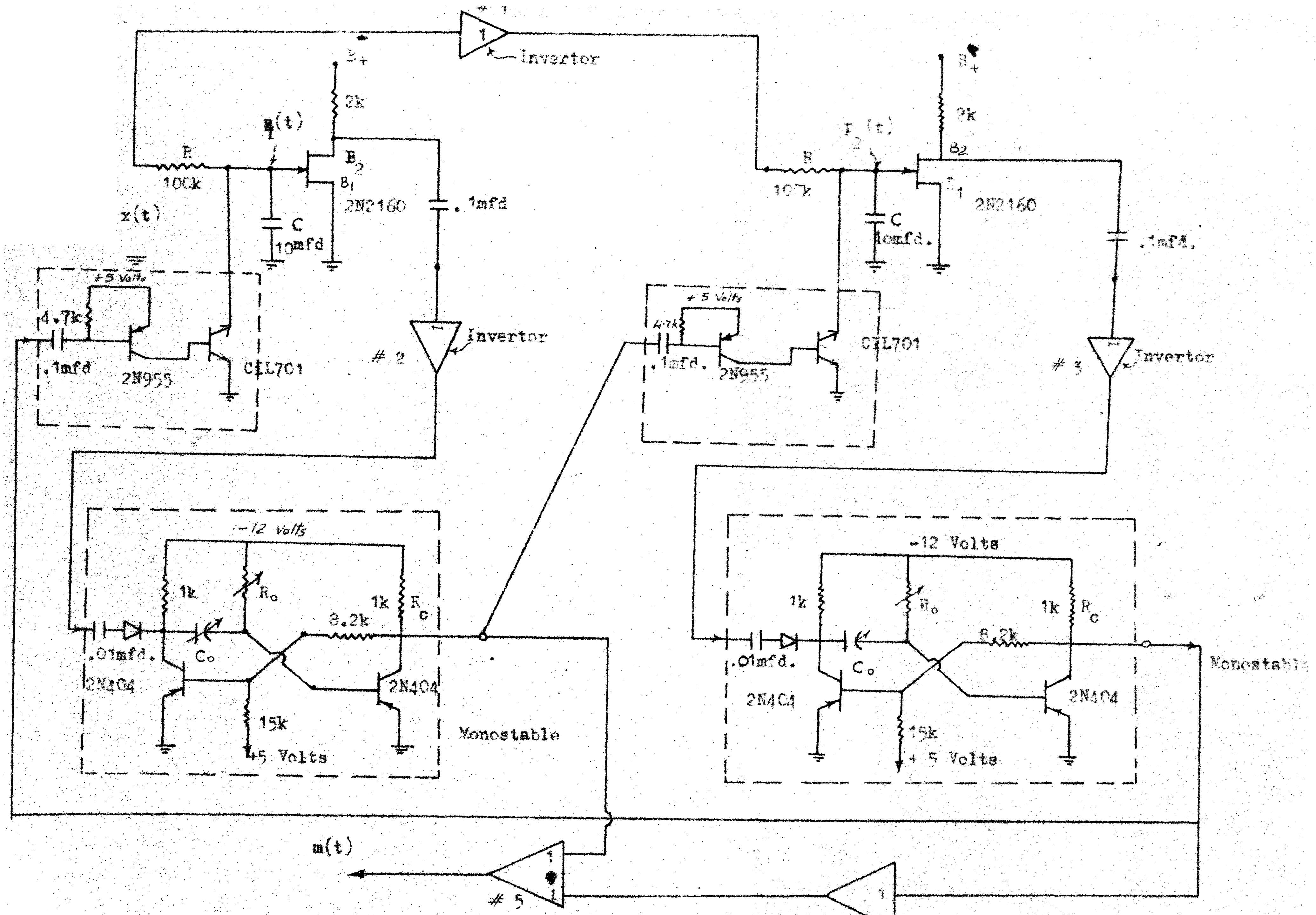
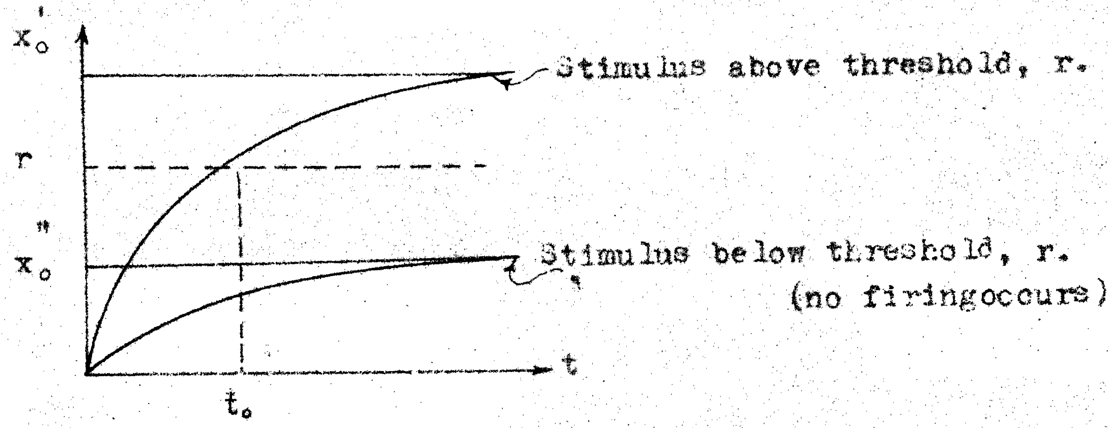
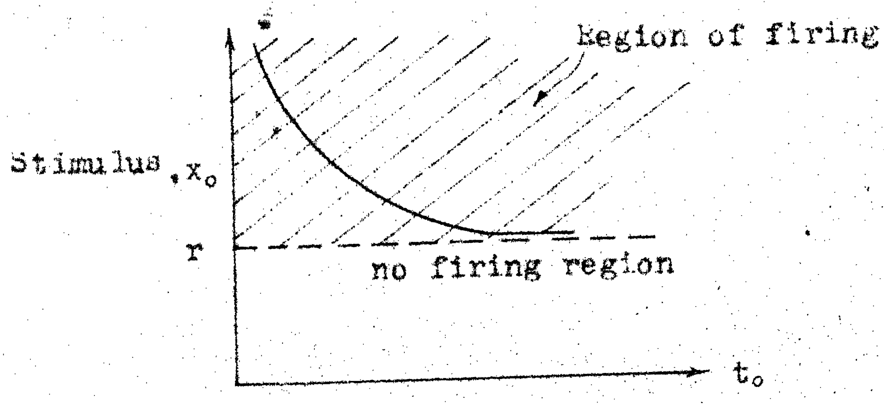

Invertor
B_+
2k
R
100k
x(t)
B_2
B_1
2N2160
C
10mfd
.1mfd
+5 Volts
4.7k
.1mfd
2N955
CIL701
2
Invertor
R
100k
B_2
2N2160
C
10mfd.
.1mfd.
+5 Volts
4.7k
.1mfd.
2N955
CIL701
3
Invertor
-12 Volts
1k
R_o
1k
R_c
8.2k
.01mfd.
C_o
2N404
2N404
15k
+5 Volts
Monostable
-12 Volts
1k
R_o
1k
R_c
8.2k
.01mfd.
C_o
2N404
2N404
15k
+ 5 Volts
Monostable
m(t)
# 5
# 4
Invertor

FIG.8



Determination of firing time t_o

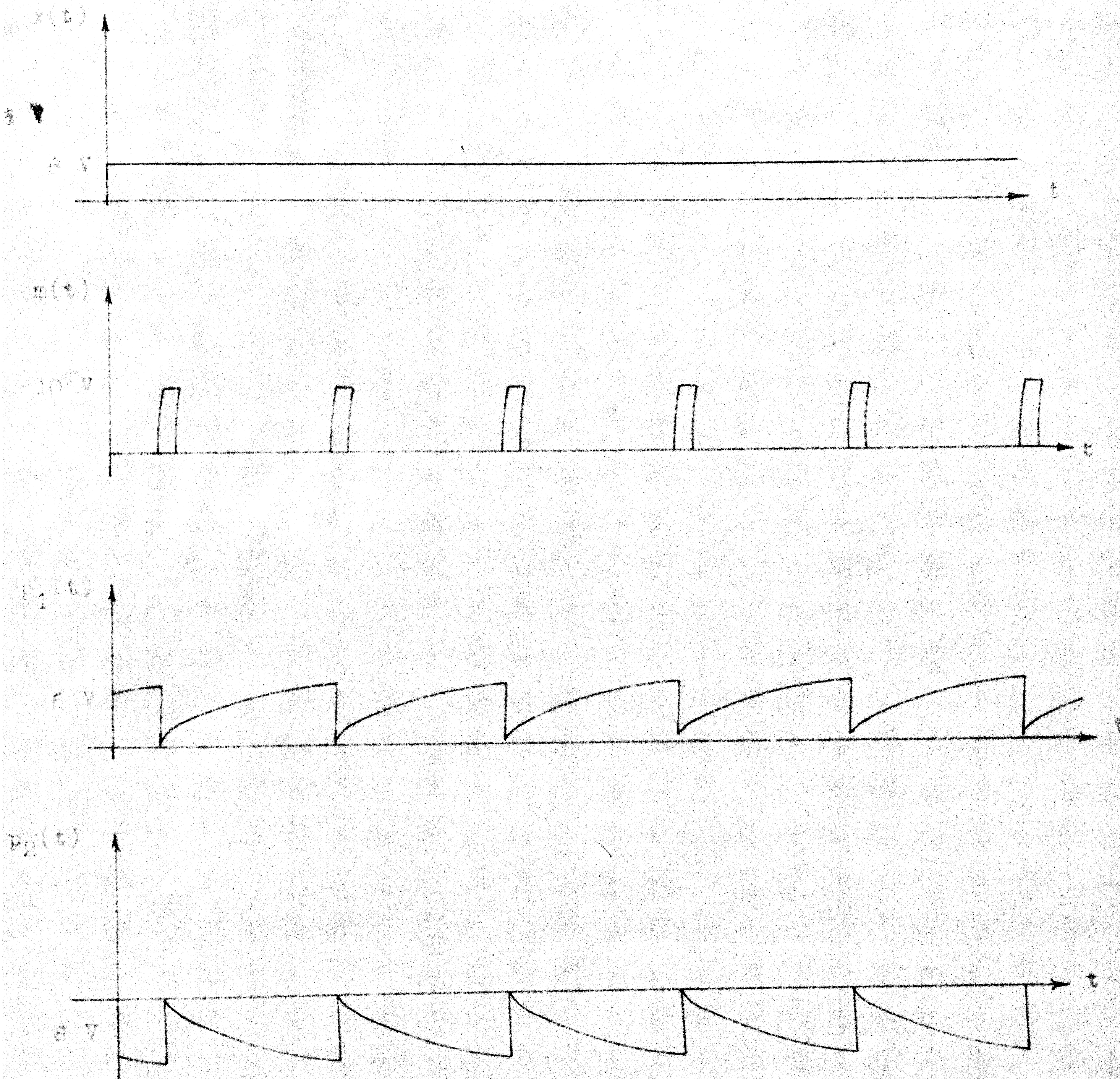
Fig. 9(a)



Strength duration curve

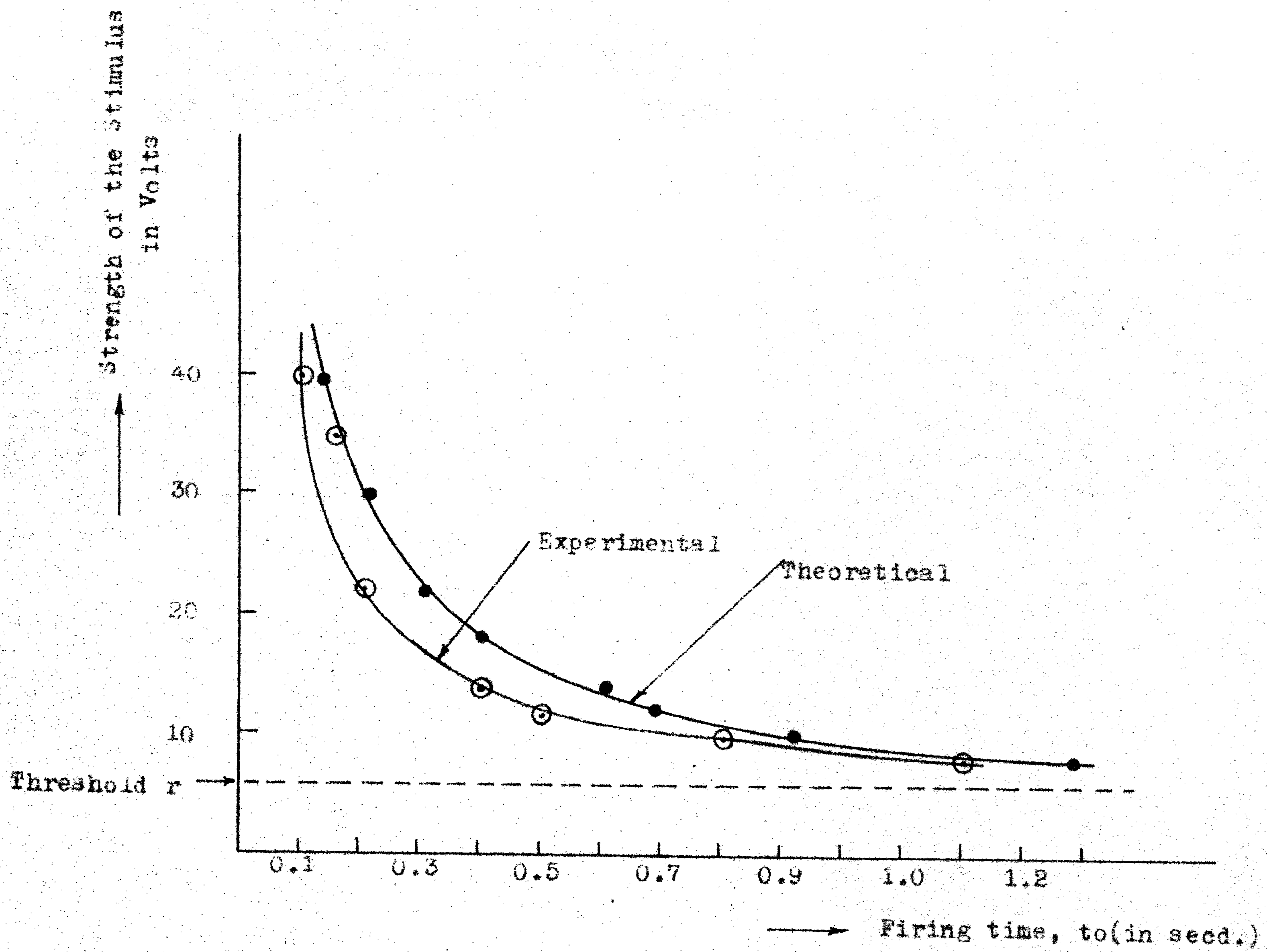
Fig. 9(b)

FIG.9



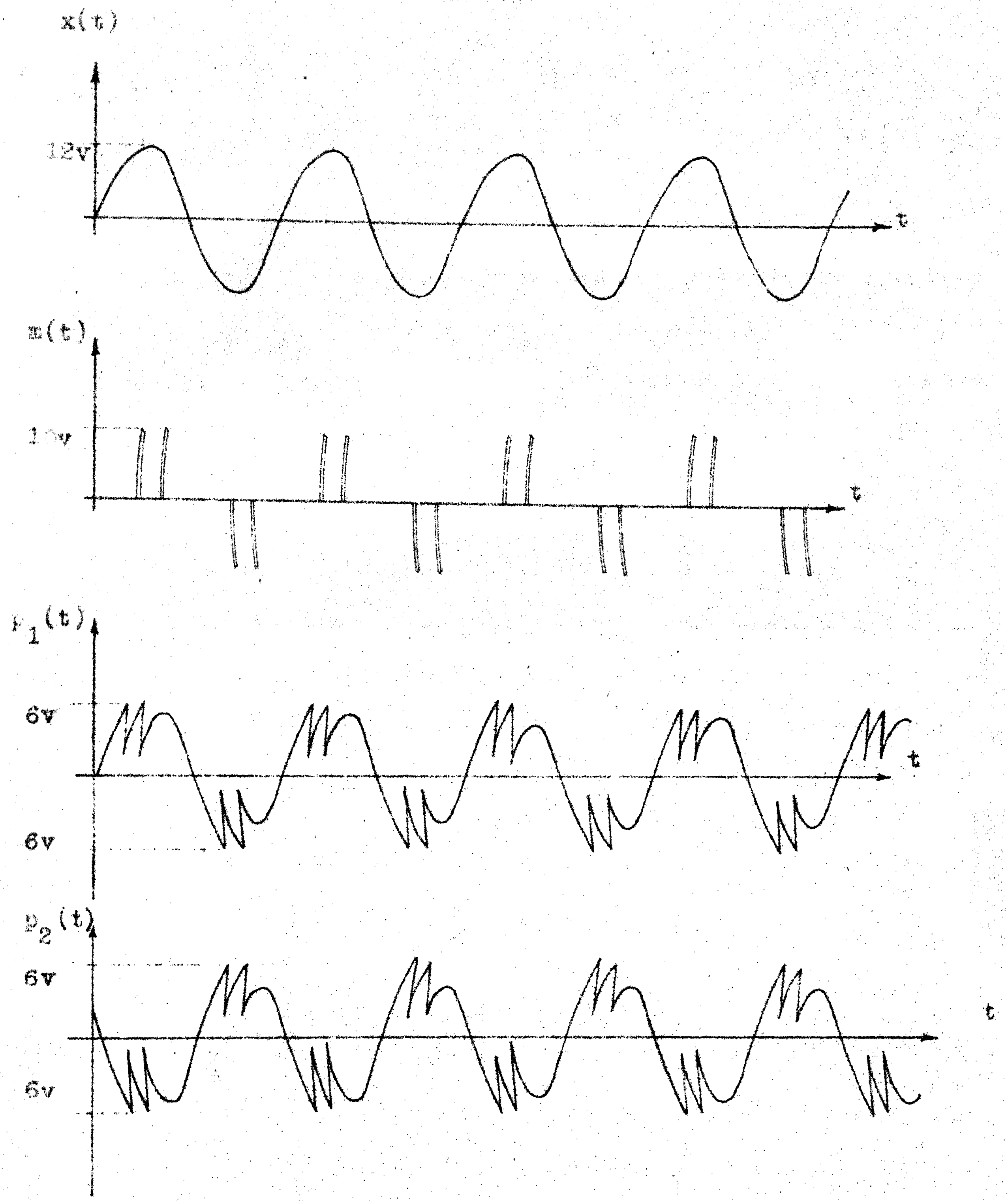
Time scale: 2.5 cm. per second

FIG. 10



Strength duration curve

FIG.11



Time scale: 0.5 cm. per second.

FIG. 12

CHAPTER - 3

LYAPUNOV'S DIRECT METHOD APPLIED TO ΣPFM FEEDBACK SYSTEMS

INTRODUCTION

This chapter deals with the application of Lyapunov's direct method to ΣPFM feedback systems. A quadratic type of Lyapunov function was considered by Pavlidis [38]. Here, simplified conditions for finding the bounds on stability sector are obtained and a few examples are worked in detail to illustrate the method.

3.1 Description of System [38,40,45]

Consider a unity feedback system with input u and output y. Sigma pulse frequency modulator is used as a controller. The block diagram is shown in Fig. 13. The state equations of this system may be written as

$$\underline{\dot{x}} = \underline{A}\ \underline{x} + \underline{b}\ m(t) \tag{3.1.1}$$

and $$\dot{p} = -\ g(p) - \underline{c}^T\underline{x} - r\ \mathrm{sgn}(p)\,\delta(|p| - r) \tag{3.1.2}$$

where $$m(t) = \mathrm{sgn}(p)\,\delta(|p| - r) \tag{3.1.3}$$

and $$\underline{c}^T\underline{x} = y - u \tag{3.1.4}$$

where the first equation represents the controlled plant, the second one the modulator. $\underline{x}$ is a n-dimensional state vector; $\underline{A}$ is nxn matrix; $\underline{b}$ and $\underline{c}$ are n vectors; r is the threshold for the modulator; p is the output of the filter in the modulator

and $g(p)$ is a nondecreasing continuous odd function of p. We shall confine ourselves to the case where $g(p) = ap$, where a is the pole of the filter transfer function in the modulator. In the literature this ΣPFM is also referred to as Neural PFM or Relaxation PFM.

3.2 Concept of Equilibrium [38,40]

In the case of ΣPFM, the equilibrium has been defined as the condition where $\dot{\underline{x}} = \underline{o}$ and $\dot{p} = o$ and no firing occurs i.e. final solution should satisfy the equations

$$\left.\begin{aligned} \underline{A}\,\underline{x} &= \underline{o} \\ -g(p) - \underline{c}^T\underline{x} &= o \\ |p| &< r \end{aligned}\right\} \qquad (3.2.1)$$

The set of equations (3.2.1) may have a number of solutions. Let the region containing these equilibrium points be denoted by a region R in the state space. A special case of the equilibrium region R is the origin in the state space. For the existance of such an equilibrium region R following theorem has been stated by Pavlidis and Jury [40].

Theorem 1

A PFM unity feedback system with a linear plant having transfer function G(s) and a gain element K (see Fig.13) has an equilibrium position for all values of input and any initial condition, defined by no pulse firing, constant output and constant value of p(t) if and only if the limit for $s \to o$ of sG(s) exists, it is different than zero and satisfies the

inequality $2g(r) \geq K \lim_{s \to o} sG(s)$ (3.2.2)

It should be noted that existence of nonzero limit of $sG(s)$ when $s \to o$ means that $G(s)$ has one pole at the origin and all its other poles have negative real parts.

After assuring the existence of equilibrium point or region, one has to worry about the stability of the point or region. In the discussion that follows, we are considering asymptotic stability of the origin in the large since ASIL of the origin is one of the most important topics of interest in the theory of stability. The stability of the equilibrium region in state space for this class of system has recently been discussed by Pavlidis [45,46].

3.3 Lyapunov's Direct Method Applied to ΣPFM

Before obtaining the sufficient conditions for asymptotic stability of the origin in the large for sigma PFM systems, some important definitions of stability and Lyapunov's main theorem are quickly reviewed [77].

Definition 1

An equilibrium state $\underline{x}_e$ of the system $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ is said to be stable if for each real number $\epsilon > o$ there is a real number $\delta(\epsilon, t_o) > o$ such that the inequality

$$\| \underline{x}_o - \underline{x}_e \| \leq \delta$$

implies

$$\| \phi(t; \underline{x}_o, t_o) - \underline{x}_e \| \leq \epsilon \text{ for } t \geq t_o \text{ (initial time)}$$

where $\underline{x}_o$ is the initial state at time t_o.

Definition 2

An equilibrium state $\underline{x}_e$ of the system $\dot{\underline{x}} = \underline{f}(\underline{x},t)$ is said to be asymptotically stable if 1) it is stable 2) every solution starting at a state $\underline{x}_o$ sufficiently near $\underline{x}_e$ converges to $\underline{x}_e$ as t increases indefinitely.

Definition 3

An equilibrium state $\underline{x}_e$ of the system $\dot{\underline{x}} = \underline{f}(\underline{x},t)$ is said to be asymptotically stable in the large (ASIL) if it satisfies definition 2 for any initial arbitrarily large state $\underline{x}_o$.

The stability with which we are interested is the asymptotic stability in the large. This for an autonomous system, physically means that for any initial large disturbance, system comes back to the origin as time increases indefinitely. Lyapunov's theorem gives sufficient conditions for the stability of such systems.

Theorem 2

If there exists in the whole state space a function $V(\underline{x},t)$ for the system defined by $\dot{\underline{x}} = \underline{f}(\underline{x},t)$ with $\underline{f}(\underline{o},t) = \underline{o}$ for all t, with the following properties

1) $V(\underline{x},t)$ is a scalar function.
2) $V(\underline{o},t) = o$.
3) $V(\underline{x},t)$ is a continuous function having continuous first partial, derivatives with respect to its arguments in some region about the equilibrium state.

4) $V(\underline{x},t)$ is a positive definite function.

5) It has a total derivative i.e. $\dot{V}(\underline{x},t)$ which when takes along the trajectory is negative definite.

6) $V(\underline{x},t) \to \infty$ as $\underline{x} \to \infty$.

then, the equilibrium state of the system ($\underline{o}$) is asymptotically stable in the large.

Now we shall apply Lyapunov's theorem for asymptotic stability of the origin in the large for sigma pulse frequency modulated control systems. Pavlidis [38] considered a quadratic type of Lyapunov function. Following treatment is based on this.

Consider the system description as given in Section 3.1. Rewriting equations (3.1.1) and (3.1.2),

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,\mathrm{sgn}(p)\,\delta\,(|p| - r)$$

$$\dot{p} = -\,ap - \underline{c}^T\,\underline{x} - r\,\mathrm{sgn}(p)\,\delta\,(|p| - r)$$

Considering an augmented state vector $\underline{z} = \begin{bmatrix} p \\ \underline{x} \end{bmatrix}_{(n+1)}$, we choose a scalar function (Lyapunov function) $V = \underline{z}^T\,\underline{P}\,\underline{z}$ where $\underline{P}$ is a positive definite $(n+1)x(n+1)$ matrix.

$$\underline{P} = \left[\begin{array}{c:c} 1 & \underline{q}^T \\ \hdashline \underline{q} & \underline{Q} \end{array}\right] \quad ;$$

$\underline{Q}$ is nxn positive definite matrix and $\underline{q}$ is an n vector.

So
$$V = [p \quad \underline{x}^T] \begin{bmatrix} 1 & \underline{q}^T \\ \underline{q} & \underline{Q} \end{bmatrix} \begin{bmatrix} p \\ \underline{x} \end{bmatrix}$$

$$= \underline{x}^T\,\underline{Q}\,\underline{x} + p^2 + 2p\,\underline{q}^T\,\underline{x}$$

Then we find $\dot{V}$

$$\dot{V} = 2\,\underline{x}^T\,\underline{Q}\,\dot{\underline{x}} + 2\,p\,\dot{p} + 2\,p\,\underline{q}^T\,\dot{\underline{x}}$$

$$= 2\,\underline{x}^T\,\underline{Q}\,\underline{A}\,\underline{x} + 2\,\underline{x}^T\,\underline{Q}\,\underline{b}\,m(t) + 2\,p(-a\,p - \underline{c}^T\,\underline{x} - r\,m(t))$$
$$+ 2\,p\,\underline{q}^T\,\underline{A}\,\underline{x} + 2\,p\,\underline{q}^T\underline{b}\,m(t) + 2\,\underline{q}^T\underline{x}(-\,a\,p - \underline{c}^T\underline{x} - r\,m(t))$$

$$= -2(V_1 + V_2\,\delta(|p| - r))$$

where

$$V_1 = -\underline{x}^T(\underline{Q}\,\underline{A} - \underline{q}\,\underline{c}^T)\,\underline{x} + p(\underline{c}^T - \underline{q}^T\,\underline{A})\,\underline{x} + a\,p\,\underline{q}^T\underline{x} + a\,p^2 \qquad (3.3.1)$$

and $$V_2 = -\underline{x}^T(\underline{Q}\,\underline{b} - r\,\underline{q})\,\mathrm{sgn}(p) + p\,(r - \underline{q}^T\,\underline{b}) \qquad (3.3.2)$$

V_1 can be written as

$$V_1 = -\underline{x}^T(\underline{Q}\,\underline{A} - \underline{q}\,\underline{c}^T)\,\underline{x} + p(\underline{c}^T - \underline{q}^T\,\underline{A} + a\,\underline{q}^T)\,\underline{x} + a\,p^2$$

$$= \underline{x}^T\,\underline{\Delta}\,\underline{x} + 2\,p\,\underline{f}^T\,\underline{x} + p^2 a$$

$$= [p \quad \underline{x}^T] \begin{bmatrix} a & \vdots & \underline{f}^T \\ \cdots & \cdots & \cdots \\ \underline{f} & \vdots & \underline{\Delta} \end{bmatrix} \begin{bmatrix} p \\ \\ \underline{x} \end{bmatrix}$$

$$= \underline{z}^T\,\underline{H}\,\underline{z}$$

where $$\underline{\Delta} = \underline{q}\,\underline{c}^T - \underline{Q}\,\underline{A} \qquad (3.3.3)$$

$$2\,\underline{f}^T = (\underline{c}^T - \underline{q}^T\,\underline{A} + a\,\underline{q}^T) \qquad (3.3.4)$$

$$\underline{H} = \begin{bmatrix} a & \vdots & \underline{f}^T \\ \cdots & \cdots & \cdots \\ \underline{f} & \vdots & \underline{\Delta} \end{bmatrix} \qquad (3.3.5)$$

<u>Theorem 3</u> [38,45]

The origin (provided it exists) of the system described by equations (3.1.1) and (3.1.2) is ASIL if the V-function (positive

definite scalar function) is constant or decreasing along the trajectory of the system when no pulses are fired and decreasing during pulse emission.

or

The origin (provided it satisfies theorem 1) of the system described by equations (3.1.1) and (3.1.2) is ASIL if the matrix $\underline{H}$ defined by eqn.(3.3.5) is semi-positive definite and V_2 defined by equation (3.3.2) is a positive definite function during the pulse emission period (i.e. when $|p| = r$).

3.4 Examples

We shall work out a few examples to illustrate the method and results will be compared to that of Popov's method and analog simulation in the next chapter. The first example is also considered by Pavlidis [38].

Example 1

Consider a plant $K\,G(s) = \frac{K}{s}$

$u \equiv o, \quad \underline{c} = 1, \quad \underline{A} = o, \quad \underline{b} = K$

To satisfy condition (3.2.2),

$$2\,a\,r \geq K \lim_{s \to o} s \times \frac{1}{s}$$

or $$K \leq 2\,a\,r \tag{3.4.1}$$

$$\underline{H} = \begin{bmatrix} a & \underline{f}^T \\ \underline{f} & \underline{\Delta} \end{bmatrix}$$

$$2\,\underline{f}^T = (\underline{c}^T - \underline{q}^T\,\underline{A} + a\,\underline{q}^T) \qquad \text{from} \quad (3.3.4)$$

and $$\triangle = (\underline{q}\,\underline{c}^T - \underline{Q}\,\underline{A}) \qquad \text{from} \quad (3.3.3)$$

on substituting the values of $\underline{c}$ and $\underline{A}$

$\underline{f} = \frac{1}{2}(1 + a\,q)$; $\triangle = q$

Hence, $$\underline{H} = \begin{bmatrix} a & \frac{1}{2}(1 + a\,q) \\ \frac{1}{2}(1 + a\,q) & q \end{bmatrix}$$

For $\underline{H}$ to be semi-positive definite

$$a \geq o \qquad (3.4.2)$$

and $$(a\,q - \tfrac{1}{4}(1 + a\,q)^2) \geq o \qquad (3.4.3)$$

Equation (3.4.3) can be satisfied only when $q = \frac{1}{a}$

Now consider V_2 during pulse emission

$$V_2 = -\underline{x}^T(\underline{Q}\,\underline{b} - r\,\underline{q})\ \text{sgn}(p) + |p|\,(r - \underline{q}^T\,\underline{b})$$

after substituting the values of $\underline{b}$, $\underline{q}$, $|p|$, we have

$$V_2 = -x(Q\,K - \frac{r}{a})\ \text{sgn}(p) + r(r - q\,K)$$

$$= x(\frac{r}{a} - Q\,K)\ \text{sgn}(p) + r(r - \frac{K}{a})$$

Now, we will prove that $x\ \text{sgn}(p) < o$ or more generally $p\,\underline{c}^T\,\underline{x} < o$ for $|p| = r$. By multiplying both sides of equation (3.1.2) with p and rearranging the terms will obtain

$$p\,\underline{c}^T\,\underline{x} = -(a\,p^2 + p\,\dot{p} + r\,|p|\,\delta(|p| - r))$$

From the elementary definition of a modulator [38] it is easy to show that $p\,\dot{p} > o$ when $|p| = r - \epsilon$ for ϵ sufficiently small. Other terms in the bracket are obviously greater than zero. Hence $p\,\underline{c}^T\,\underline{x} < o$.

Hence for V_2 to be positive definite,

$$\left(\frac{r}{a} - Q\,K\right) < o \qquad \text{and} \qquad \left(r - \frac{K}{a}\right) > o$$

For this, gain K satisfies the conditions

$$\frac{r}{a\,Q} < K < a\,r \tag{3.4.4}$$

Q can be chosen infinitely large so that the lower bound on K becomes zero and hence $K < a\,r$.

So PFM feedback system having a first order integral plant is stable (origin is ASIL) for all gain less than a r.

Example 2

Consider a plant $K\,G(s) = \dfrac{K}{s(s+1)}$

So, $\underline{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$; $\underline{b} = \begin{bmatrix} o \\ K \end{bmatrix}$; $\underline{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u = o, \quad g(p) = a\,p, \quad a > o$

To satisfy condition (3.2.2),

$$2\,a\,r \geq K \lim_{s \to o} \frac{s}{s(s+1)} \qquad \text{or} \quad K < 2\,a\,r \tag{3.4.5}$$

Now, assume $\underline{Q} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$; $\underline{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

Since $2\,\underline{f}^T = (\underline{c}^T - \underline{q}^T\underline{A} + a\,\underline{q}^T)$ from (3.3.4)

So $\underline{f}^T = \left(\dfrac{1 + a\,q_1}{2}, \dfrac{(q_2 - q_1 + a\,q_2)}{2} \right)$

and $\underline{\Delta} = \underline{q}\,\underline{c}^T - \underline{Q}\,\underline{A}$ from (3.3.3)

$$= \begin{bmatrix} q_1 & (Q_2 - Q_1) \\ q_2 & (Q_4 - Q_3) \end{bmatrix}$$

$$\underline{H} = \begin{bmatrix} a & \underline{f}^T \\ \underline{f} & \underline{\Delta} \end{bmatrix}$$

$$= \begin{bmatrix} a & \tfrac{1}{2}(1+aq_1) & \tfrac{1}{2}(q_2-q_1+aq_2) \\ \tfrac{1}{2}(1+aq_1) & q_1 & (Q_2-Q_1) \\ \tfrac{1}{2}(q_2-q_1+aq_2) & q_2 & (Q_4-Q_3) \end{bmatrix}$$

Now, for $\underline{H}$ to be semi-positive definite

$$a \geq o \qquad (3.4.6)$$

$$(a\,q_1 - \tfrac{1}{4}(1 + a\,q_1)^2) \geq o \qquad (3.4.7)$$

and $\quad \det.\,\underline{H} \geq o \qquad (3.4.8)$

Condition (3.4.6) is obviously satisfied as $a > o$, condition (3.4.7) can be satisfied only as equality and the possible choice of $q_1 = \dfrac{1}{a}$ and condition (3.4.8) gives

$$\det. \begin{bmatrix} a & 1 & t \\ 1 & \frac{1}{a} & (Q_2-Q_1) \\ t & q_2 & (Q_4-Q_3) \end{bmatrix} \geq o \quad , \quad \text{where } t = \tfrac{1}{2}(q_2 - \frac{1}{a} + aq_2)$$

$$\text{or} \quad (a(\frac{1}{a}(Q_4-Q_3) - q_2(Q_2-Q_1)) - ((Q_4-Q_3) - q_2 t) + t((Q_2-Q_1) - \frac{t}{a})) \geq o$$

$$\text{or} \quad (aq_2(Q_1-Q_2) + q_2 t + t(Q_2-Q_1) - \frac{t^2}{a}) \geq o$$

$$\text{or} \quad ((Q_1-Q_2)(aq_2-t) + t(q_2 - \frac{t}{a})) \geq o \qquad (3.4.9)$$

It is possible to satisfy the condition (3.4.9) as inequality by the following choice of parameters.

Q_1 and Q_2 are positive numbers and $Q_1 > Q_2$

$t = 0$ and q_2 be a positive quantity.

$$t = 0 \text{ gives } \tfrac{1}{2}(q_2 - \frac{1}{a} + a\, q_2) = o$$

$$\text{or} \quad q_2(1 + a) = \frac{1}{a}$$

$$\text{or} \quad q_2 = \frac{1}{a(1 + a)} \qquad (3.4.10)$$

Since $a > o$, q_2 is a positive number.

Now consider V_2 during pulse emission,

$$V_2 = - \underline{x}^T(\underline{Q}\ \underline{b} - r\ \underline{q})\ \text{sgn}(p) + |p|(r - \underline{q}^T\ \underline{b})$$

$$= [x_1 \quad x_2] \begin{bmatrix} (rq_1 - KQ_2) \\ (rq_2 - KQ_4) \end{bmatrix} \text{sgn}(p) + r(r - Kq_2)$$

$$= x_1\ (r\ q_1 - K\ Q_2)\ \text{sgn}\ (p) + x_2(r\ q_2 - KQ_4)\text{sgn}(p) + r(r-Kq_2)$$

Substituting $q_1 = \frac{1}{a}$ and $q_2 = \frac{1}{a(1+a)}$

$$V_2 = x_1\left(\frac{r}{a} - K\,Q_2\right)\operatorname{sgn}(p) + x_2\left(\frac{r}{a(1+a)} - K\,Q_4\right)\operatorname{sgn}(p) + r\left(r - \frac{K}{a(1+a)}\right)$$

It has been shown in the previous example that

$$p\,\underline{c}^T\,\underline{x} < 0 \qquad \text{or} \qquad x_1 \operatorname{sgn}(p) < 0$$

Now we get following conditions to make V_2 strictly positive during pulse emission

$$\left(\frac{r}{a} - K\,Q_2\right) < 0 \tag{3.4.11}$$

$$\left(\frac{r}{a(a+1)} - K\,Q_4\right) = 0 \tag{3.4.12}$$

$$\left(r - \frac{K}{a(a+1)}\right) > 0 \tag{3.4.13}$$

Condition (3.4.11) gives $K > \frac{r}{a\,Q_2}$ (3.4.14)

This provides a lower bound on K equal to zero by a very large choice of Q_2.

Condition (3.4.12) gives

$$Q_4 = \frac{r}{K\,a(a+1)} \tag{3.4.15}$$

Since the choice of Q_4 is an arbitrary positive number, it is always possible to satisfy (3.4.15).

Condition (3.4.13) gives

$$(r\,a(a+1) - K) > 0$$

$$\text{or} \qquad K < a\,r(a+1) \tag{3.4.16}$$

Condition (3.4.16) gives an upper bound on gain K.

Hence, finally we get the condition (3.4.16) with the following choice of arbitrary parameters.

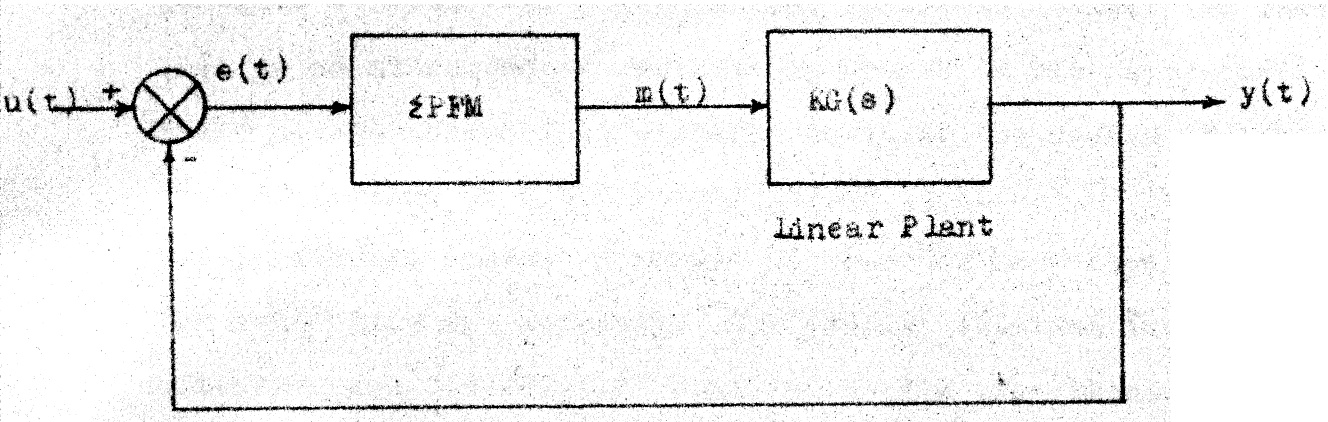
$q_1 = \frac{1}{a}$; $q_2 = \frac{1}{a(a+1)}$; Q_1 and Q_2 are positive and $Q_1 > Q_2$; Q_4 is a positive number satisfying (3.4.15) and $Q_3 = 0$ so as to ensure positive definiteness of $\underline{Q}$.

Stability sector K is decided by the conditions (3.4.5) and (3.4.16).

For the case when $a = 1$, we get $K < 2\,r$.

3.5 Conclusions

Besides the origin as the equilibrium point, PFM feedback system has been observed to have a region of equilibrium points, say $\mathcal{R}$. After initial disturbance system comes back to $\mathcal{R}$ also known as subthreshold region where $|p| < r$. Hence stability studies of equilibrium region $\mathcal{R}$ is of equally great interest. Recently a few papers have appeared for such analysis by Pavlidis [45,46].



A ΣPFM feedback System

FIG.13

CHAPTER - 4

ABSOLUTE STABILITY OF ΣPFM SYSTEM THROUGH POPOV'S CRITERION

INTRODUCTION

The present chapter is devoted to the absolute stability of sigma pulse frequency systems using Popov's method. A generalised Popov's criterion has been formulated to investigate the absolute stability of principal and particular cases. In fact, this is an extension of the work by Dymkov in his paper [54], where he considered the principal cases by the choice of $q = o$. The restriction on q has been modified to be $q \leqslant o$ for including the particular cases. A number of examples have been worked out for validating the approach. The results obtained from Popov's criterion are conservative compared to the experimental results. This is so because of the restriction on the parameter q and the inherent approximations.

4.1 A Short Review of Popov's Criterion for Absolute Stability

Absolute Stability:

Let us consider the system

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{b}\,y \\ y &= \phi(\sigma) \\ \sigma &= \underline{c}^T \underline{x} \end{aligned} \tag{4.1.1}$$

where $\underline{x}$ is a n-dimensional state vector, $\underline{A}$ is a nxn matrix, $\underline{b}$ and $\underline{c}$ are n-dimensional vectors, $\phi(\sigma)$ is a memoryless time-invariant nonlinearity. The block diagram representation of (4.1.1) is shown in Fig.14(a). G(s) is the linear part of the system.

Restrictions on nonlinearity $\phi(\sigma)$:

(i) $\phi(\sigma)$ is continuous; $\phi(\sigma) = 0$ for $\sigma = 0$

(ii) Consider $0 \leq \phi(\sigma) \leq k$, denoted as sector $[0,k]$ i.e. $\phi(\sigma)$ lies inside the first and third quadrants in the sector formed by the σ- axis and the line $y = k\sigma$. See Fig.14(b). This type of sector is considered for the principal case when matrix $\underline{A}$ is Hurwitz i.e. all eigenvalues of $\underline{A}$ have negative real parts.

(iii) Consider $0 < \phi(\sigma) \leq k$, denoted as sector $(0,k]$, for the simplest particular case when $\underline{A}$ has a zero eigenvalue of multiplicity one.

Now, we shall define the absolute stability. If the origin of the system described by (4.1.1) is asymptotically stable in the large for all nonlinearities satisfying the above stated restrictions we say that the system is absolutely stable. Certain theorems due to Popov are now reviewed for absolute stability.

The V.M. Popov Theorem:

For the system described by (4.1.1) to be absolutely stable in the sector $[0,k]$ for the principal case, and in the sector$[\epsilon,k]$ for the particular cases (where $\epsilon > 0$ is an arbitrarily small number), it is sufficient that there exist a finite real number q such that for all $w \geq 0$ the following inequality is satisfied

$$\operatorname{Re}(1 + jqw)\, G(jw) + \frac{1}{k} > 0$$

and, additionally for the particular cases, that the conditions for stability in the limit be satisfied.

Conditions for Stability in the Limit:

For a particular case of (4.1.1) to be stable in the limit, it is necessary and sufficient that each root on the imaginary axis, jw_o, (w_o is a real number) of its characteristic equation have a multiplicity γ_o not greater than two, and that the following conditions be satisfied:

a) for $w_o = o$, $\gamma_o = 1$:

$$\lim_{w \to +o} \operatorname{Im} G(jw) = -\infty \quad \text{or} \quad \operatorname{Im} G^*(o) < o \qquad \text{where}$$

$G^*(w)$ is the modified frequency function and is related to $G(jw)$ by the equations

$$\operatorname{Re} G^*(w) = \operatorname{Re} G(jw)$$

$$\operatorname{Im} G^*(w) = w \operatorname{Im} G(jw)$$

b) for $w_o \neq o$, $\gamma_o = 1$: when w traverses the point w_o going from values of $w < w_o$ to values of $w > w_o$, the frequency response $G(jw)$ (or $G^*(w)$) goes to a point at infinity in such a way that points on the negative real axis with abscissas of arbitrarily large magnitude (i.e., which are close to $-\infty$) remain to the left of it;

c) for $w_o = o$, $\gamma_o = 2$:

$\lim \operatorname{Re} G(jw) = -\infty$, and $\operatorname{Im} G(jw) < o$ for small values of $w > o$ (or identical conditions for $G^*(jw)$;

d) for $w_o \neq o$, $\gamma_o = 2$:

$$\lim_{w \to w_o - o} \operatorname{Re} G(jw) = \lim_{w \to w_o + o} \operatorname{Re} G(jw) = -\infty$$

and

$$\operatorname{Im} G(jw) \begin{cases} o & \text{for } w_o - \eta < w < w_o \\ o & \text{for } w_o < w < w_o + \eta \end{cases} \qquad (\eta > o \quad \text{small})$$

(or identical conditions for $G^*(w)$).

The case $k = \infty$:

1) The principal case of the system (4.1.1) is absolutely stable in the sector $[o,\infty]$ if

$$\text{Re } (1 + jqw)\, G(jw) > o,$$

for some $q \geq o$ and all $w \geq o$, and in addition

$\lim_{w \to \infty} jw\, G(jw) \neq o$ which is equivalent to the condition that difference of order of numerator and denominator of G(jw) is restricted to unity, i.e. $n-m = 1$ where n is the order of the denominator, m is the order of numerator.

2) The simplest particular case of the system (4.1.1) is absolutely stable in the sector $(o,\infty]$ if

for some q and all $w > o$, the weakened Popov condition

$$\text{Re } (1 + jqw)\, G(jw) \geq o,$$

is satisfied and in addition, the condition for stability in the limit $\text{Im } G^*(o) < o$, is satisfied.

4.2 Popov Criterion Applied to ΣPFM Systems

Consider the PFM control system shown in Fig.15(a). $W_B(s)$ is the transfer function of the filter which characterises the kind of PF modulation, γ is the level which, when reached, initiates the return of the signal at the output of $W_B(s)$ to zero, $a = \lim_{s \to \infty} s\, W_B(s)$, x(t) is the error input to the modulator, m(t) is the output of the modulator and y(t) is the output of the plant.

It may be noted that $W_B(s) = \frac{a}{a+s}$ represents the filter for PFM; for IPFM, we have to consider $W_B(s)$ of the form $\frac{1}{s}$. The nonlinearity $\phi(p)$ is a single-relay hysteresis nonlinearity. Further restrictions on $\phi(p)$ are as follows:

i) The curve $\phi(p)$ is single valued at the point $p = o$ and $\phi(p) = o$ when $p = o$;

ii) There exists a range of values $p \neq o$ ($|p| < r$) for which $\phi(p) = o$;

iii) The condition $o \leq \frac{\phi(p)}{p} < \infty$ is satisfied

i.e. $o \leq \frac{\phi(p)}{p} < \infty$;

iv) The hysteresis characteristics $\phi(p)$ does not comply with the positiveness condition [80]. The hysteresis function satisfies the condition of positiveness if for any absolutely continuous function $p(t)$ and $\phi_o \in E(p(o))$ there exists a constant $\gamma \geq o$ such that $\int_o^t \phi(p, \phi_o)_\tau \, d\, p(\tau) \geq -\gamma$ when $t \geq o$.

The condition of positiveness physically means that the net area enclosed by the loop is positive.

Fig.15 has been reduced to the form Fig.16.

$$W_L(s) = (G(s) + \frac{r}{a})\, W_B(s) \tag{4.2.1}$$

Principal Case 54 :

When $W_L(s)$ has all the poles in the left-hand half plane, then the system shown in Fig. 16 is absolutely stable in the sector $[o, \infty]$ if

$$\mathrm{Re}\ (1 + jqw)\ W_L(jw) > 0 \quad \text{for some } q \geq 0 \text{ and all } w \geq 0$$

And for the cases of hysteresis nonlinearities that do not comply with the condition of positiveness [80], $q \leq 0$

So consequently for

$$\mathrm{Re}(1 + jqw)\ W_L(jw) > 0 \quad \text{to hold}$$

$$q = 0$$

This gives $\mathrm{Re}\ W_L(jw) > 0$ (4.2.2)

Simplest Particular Case:

The simplest particular case of the system shown in Fig.16 is absolutely stable in the sector $(0, \infty]$ if for some q and all $w \geq 0$, the weakened Popov condition

$\mathrm{Re}(1 + jqw)\ W_L(jw) \geq 0$, is satisfied and in addition, the condition for stability in the limit $\mathrm{Im}\ W_L^*(0) < 0$, is satisfied. And for the cases of hysteresis nonlinearities that do not comply with the condition of positiveness, $q \leq 0$, hence for absolute stability of simplest particular case in the sector $(0, \infty]$, ome must satisfy

$$\mathrm{Re}(1 + jqw)\ W_L(jw) \geq 0 \quad \text{for} \quad q \leq 0 \quad \text{and all} \quad w \geq 0 \quad \text{and}$$

$$\mathrm{Im}_{w \to 0}\ wW_L(jw) < 0$$

4.3 Examples

Ex.1:

Consider $G(s) = \frac{K}{s+1}$; K is the linear gain of the plant.

$W_B(s) = \frac{1}{s+1}$; $a = 1$

$$W_L(s) = \frac{1}{s+1}\left(\frac{K}{s+1} + r\right) = \frac{K + r(s+1)}{(s+1)^2} .$$

For absolute stability, we have to satisfy the inequality

$$\mathrm{Re}\, W_L(jw) > o$$

or

$$\mathrm{Re} \left(\frac{K + r(jw + 1)}{(jw + 1)^2} \right) > o$$

or

$$K + r - Kw^2 + rw^2 > o$$

or

$$K(1 - w^2) + r(1 + w^2) > o$$

or

$$K(w^2 - 1) < r(1 + w^2)$$

For $o \leq w \leq 1$, $K >$ a negative quantity and

for $1 \leq w \leq \infty$, $K < r$ hence,

for all $w \geq o$, $K < r$

So system is absolutely stable for all the gain strictly less than the threshold r i.e. origin is ASIL for all K less than threshold for the nonlinearities in the sector $[o, \infty]$.

This example shows the application of the theorem given by Dymkov [54] for the cases when $W_L(s)$ has all the poles in the left half plane.

Ex.2:

Consider $G(s) = \frac{K}{s}$

$$W_B(s) = \frac{a}{s+a}$$

So

$$W_L(s) = \left(G(s) + \frac{r}{a}\right) \frac{a}{s+a} = \left(\frac{aK + rs}{s(s + a)} \right)$$

$W_L(s)$ has one pole at the origin, hence this corresponds to simplest particular case. So that the origin be ASIL for the sector $(o, \infty]$, we must satisfy

$$\lim_{w \to o} w\, W_L(jw) < o \qquad \text{(stability in the limit)}$$

and $\quad Re(1 + jqw)\, W_L(jw) \geq o \quad$ for $\; q \leq o \;$ and $\;$ all $w \geq o$

(weakened Popov condition)

now $\quad W_L(jw) = \dfrac{(aK + jrw)}{jw(jw + a)}$

$$\lim_{w \to o} w\, W_L(jw) = \underset{w \to o}{\text{Limit}} - \frac{(a^2k + rw^2)}{w^2 + a^2} = \text{a negative quantity}$$

since $K > o$. Hence, stability in the limit is satisfied.

Now $\quad Re(1 + jqw)\, W_L(jw) \geq o \quad$ gives

$$(r + aqK) - (Ka - qrw^2) \geq o$$

or $\quad r(1 + qw^2) - aK(1 - q) \geq o$

or $\quad aK \leq \dfrac{r(1 + qw^2)}{(1 - q)} \quad$ for $\quad q \leq 1$

here, restriction on q is that $q \leq o$.

For maximum sector $\; q = o$ gives

$$aK \leq r$$

for $\quad a = 1,\; K \leq r$

Ex.3:

$$G(s) = \frac{K}{s(s + 1)} ;$$

$$W_B(s) = \frac{a}{s + a} .$$

$$W_L(s) = \left(G(s) + \frac{r}{a}\right)\left(\frac{a}{s+a}\right) = \frac{aK + rs(s+1)}{s(s+1)\,(s+a)}$$

$$W_L(jw) = \frac{(aK - w^2r) + jwr}{jw(jw+1)\,(jw+a)}$$

$$\lim_{\omega \to 0} \operatorname{Im} w\, W_L(jw) = -\frac{K}{a} = \text{a negative quantity since } K > o,\ a > o$$

hence, stability in the limit is satisfied.

Now for absolute stability

$$\operatorname{Re}(1 + jqw)\, W_L(jw) \geq o \quad \text{for} \quad q \leq o \quad \text{and} \quad w \geq o$$

This gives

$$\operatorname{Re} \frac{(aK - w^2r - qw^2r) + jw(r + q(aK - w^2r))}{jw(jw + 1)\ (jw + a)} \geq o$$

or $\quad (wr + wqaK - w^3qr)\ (a-w^2) - (aK-w^2r - qw^2r)\ (w + aw) \geq o$

for a = 1, we have

$$r(1 + w^2 + qw^2 + qw^4) + K(q - qw^2 - 2) \geq o$$

Maximum sector occurs at $q = o$ for $w \geq o$ and $K_{max} = \frac{r}{2}$.

Experimental Results:

In the last chapter, we applied Liapunov's direct method for finding the bounds on the stability sector, here Popov's criterion has been applied for determining the critical sector. For the sake of comparasion, an experiment by analog simulation was carried out. The Table 2 gives an idea about the stability sectors obtained by three methods.

Table 2

Threshold r	Plant	Critical Stability Sector		
		Liapunov method	Popov's method	Experimental simulation
1.2	$\frac{K}{s}$	1.2	1.2	3
2.2		2.2	2.2	6
3.0		3.0	3.0	7
4.0		4.0	4.0	9
1.4	$\frac{K}{s(s+1)}$	2.8	0.7	4
2.0		4.0	1.0	7
3.0		6.0	1.5	8
4.2		8.4	2.1	10

4.4 Conclusions

It is obvious that the results obtained from Popov's criterion are conservative. It is so because the choice of single-relay-hysteresis nonlinearity gives a severe restriction on the parameter q. Although, it is possible to get quite an improved sector by the choice of $q \geq o$, but then it seeks for the modification on the nonlinearity.

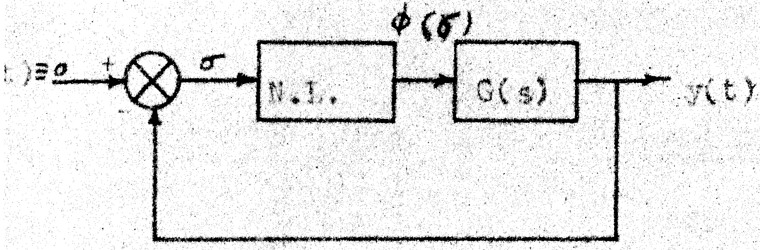


Fig.14 (a)

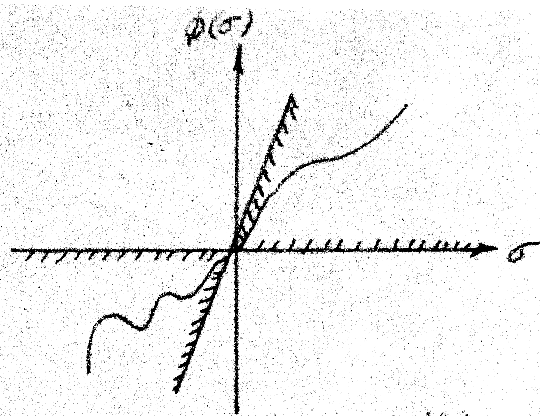
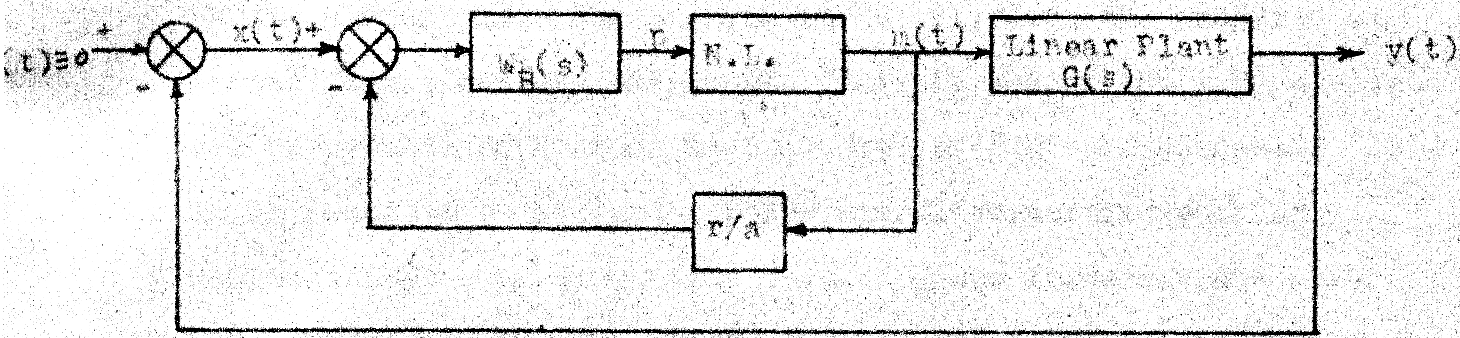


Fig.* 14 (b)

FIG.14



Zero input ≰PFM control System

Fig.15 (a)

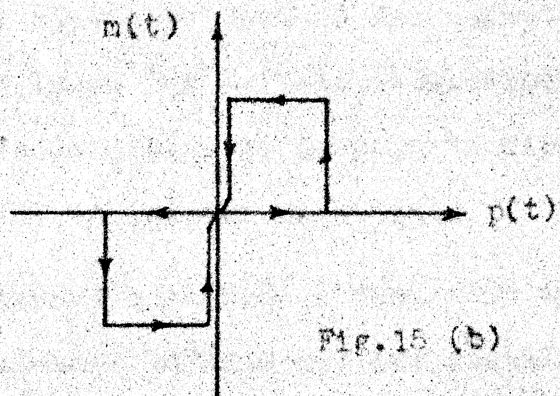
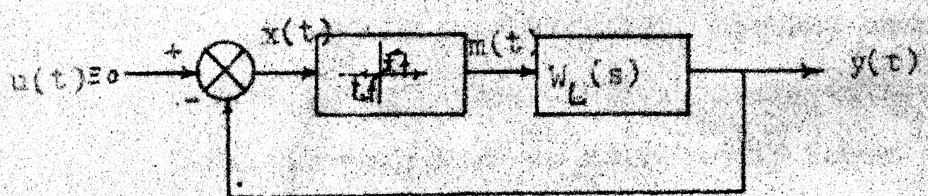


Fig.15 (b)

Nonlinearity ϕ (p)

FIG.15



Equivalent of Fig.15

FIG.16

CHAPTER - 5

CONCLUSIONS

There are several ways of implementing the circuit for sigma pulse frequency modulator. In this thesis, one such design has been implemented. This circuit makes use of two channels to respond to alternating error inputs. It is possible to have only one channel with the help of a device which breaks for both positive and negative thresholds. One of the important properties of the circuit has been studied in detail, i.e. the property of firing over a certain threshold. This is one of the main electrical properties of neurons as indicated by Jury and Blanchard [23]. It is possible to study the other neural properties such as adaptation etc. by simulating a sigma pulse frequency modulator with a time varying threshold.

The major part of this thesis has been concerned with examining the stability sectors for asymptotic stability of the origin in the large for certain closed-loop PFM systems by three different methods namely 1) Lyapunov's direct method, 2) Popov's method and 3) experimental simulation.

In Lyapunov's method, a quadratic type of V-function [38] has been considered to find out the stability boundaries of certain ΣPFM feedback systems. The V-function includes the actual system nonlinearity, without assuming any sector for the nonlinearity. Hence, the results obtained from this method are less conservative as compared to Popov's method. In Popov's

method, we have assumed that nonlinearity passes through the origin and is single valued and continuous at the origin [54]. This inherent approximation yields the results less satisfactory and more conservative. Furthermore, the condition on parameter q also contributes to yielding the less accurate results. However, the analysis in frequency domain makes this method more attractive and straightforward. It may be possible to use Popov's criterion extended for a class of nonlinearities which are discontinuous at the origin [82]. This might give a more accurate information about the stability boundary, when applied to ΣPFM systems.

An attempt has been made to give a comprehensive survey of literature including almost all the aspects of pulse frequency systems. A number of problems for future research have been mentioned. Besides its main application in control, emphasis has been laid on its wide scope in bio-medical engineering where sensory-organs of physiological systems generate pulse frequency modulated signals and transmit these signals through various nerve-fibres.

References

Variable Frequency Sampling

1. R.E. Hufnagel, "Analysis of cyclic rate sampled-data feedback control systems," Trans. AIEE, Vol.77, pp. 421-423; 1958.

2. R.E. Hufnagel, "Analysis of aperiodically sampled-data feedback control systems," Ph.D. dissertation, Corell University Ithaca, N.Y.; 1959.

3. E.I. Jury and F.J. Mullin, "The analysis of sampled-data control systems with a periodically time-varying sampling rate," IRE. Transaction on Automatic Control, Vol.AC-4, pp.15-21; No.1, May 1959.

4. R.C. Dorf, M.C. Farren and C.A.Phillips, "Adaptive sampling-frequency for sampled-data control systems," IRE transactions on Automatic Control, Vol.AC-7, Jan.1962.

5. P.M. Will, "Variable frequency sampling," Correspondence, IRE Trans. on Automatic Control, Oct. 1962.

6. S.C. Gupta, "Adaptive gain and adaptive sampling sampled data control systems", presented at IEEE Winter General meeting, New York, Jan.27-Feb.1, 1963.

7. S.C. Gupta, "Increasing the sampling efficiency for a control systems" Correspondence, IEEE Trans. on Automatic control, July 1963, Vol.AC-8, pp.263-264.

8. R. Tomovic and G.A. Bekey, "Adaptive Sampling based on amplitude sensitivity," IEEE Trans. on Automatic Control (short paper) Vol.AC-11, pp.282-284, April 1966.

9. G.A. Bekey and R. Tomovic, "Sensitivity of discrete systems to variation of sampling interval," IEEE Trans. on Automatic Control (Short Paper), Vol.AC-11,pp.284-287, April 1966.

10. P. Vidal and F. Laurent, "Asymptotic Stability of sampled-data nonlinear systems with variable sampling periods - a bionic application," Presented at the 3rd IFAC Congress London, 1966 (paper 30 c).

11. D. Ciscato and L. Mariani, "On increasing sampling efficiency by adaptive sampling," (Correspondence paper) IEEE Trans. on Automatic Control, Vol.AC-12, No.3, June 1967.

Motivation for Study of PFM Systems

(a) Neuron modelling:

12. A.L. Hodgkin and A.F. Huksley, "A quantitative description of membrane currents and its application to conduction and excitation in nerve," Journal of Physiology, 117, pp.500-544, March 1952.

13. L.D. Harmon, "Artificial Neuron", Science 129, April 1959.

14. M.H. Goldstein, Jr., "A statistical model for interpreting neuroelectric responses," No.1, March 1960, Information and Control.

15. Jones, et al., "Pulse modulation in physiological systems, phenomenological aspects," IRE Trans. on Bio-Medical Engineering, January 1961.

16. F.F. Hiltz, "Analog computer simulation of a neural element," IRE Trans. on Bio-Medical Engineering, January 1962.

17. E.I. Jury and T. Pavlidis, "A literature survey of bio-control systems," IEEE Transaction on Automatic Control, July 1963.

18. T. Pavlidis, "A new model for single neural nets and its application in the design of a neural oscillator," Bulletin Math. Biophysics, Vol.27, June 1965.

19. P.G. Caumon, "A neuron modelling," M.S. Project, University of California at Berkeley, June 1965.

20. E.A. Iyubinskii and N.V. Poziw, "Modelling information processing by neurons," Automatic and Telemechanics, Vol.26, No.10, pp.1746-1756, Oct. 1965, Moscow, U.S.S.R.

21. J.G. Blanchard, "Theory and applications of pulse-frequency modulated systems", Ph.D. Thesis, Department of Electrical Engineering, University of California at Berkeley, January 1966.

22. T. Pavlidis, "Design of neural nets with intermittent response and certain other relevant studies," Bull.Math.Biophysics, Vol.28, 1966, pp.51-74.

23. E.I. Jury and J.G. Blanchard, "On the neuron modelling," Electronics Research Lab., College of Engineering, University of California, Berkeley, Memo. ERL-M172, Aug.1, 1966.

24. T. Pavlidis, "The relation between a certain class of abstract systems and models for neural nets," (Correspondence paper), Vol.AC-12, No.1, Feb. 1967.

(b) Use of PFM in Adaptive Control:

25. G. Murphy and K.L. West, "The use of pulse-frequency modulation for adaptive control," Proceedings of NEC, Chicago, Ill. Vol.18, pp.271-277, Oct.8, 1962.

(c) Use of PFM in Incremental Servos:

26. S.J. Bailey, "Incremental Servos," Control Engineering, Jan. 1961.

27. R.B. Kieburtz, "The step motor - the next advance in control systems," January 1964, IEEE Trans. on Automatic control.

28. R.C. Dorf and T. Liang, "A time optimal pulse frequency modulated step motor servo system," Proc. Natural Electronics, Conf. (Chicago, Ill. Oct. 1966).

Classification of PFM Systems, Methods of Stability Analysis and Optimization of PFM Control Systems.

29. A.E. Ross, "Theoretical study of pulse frequency modulation," Trans. IRE Vol.37, pp.1277-1286, Nov.1949.

30. Y.I. Khurgin, "A class of random impulse processes," Radiotekhn, Electron 2, No.4, 1957.

31. K. Izawa, "Discontinuous feedback control systems with sampling action," First IFAC Congress Moscow, Russia, 1960.

32. C.C. Li, "Integral pulse frequency modulated control systems," Ph.D. dissertation, Northwestern University, Evanston, Ill. June 1961.

33. A.U.Meyer, "Pulse frequency modulation and its effects in feedback control systems", Ph.D. dissertation, Northwestern University, Evanston, Ill., August 1961.

34. R.W. Jones, C.C. Li, A.U. Meyer and R.B.Pinter, "Pulse Modulation in physiological systems, phenomenological aspects," IRE Trans. Bio-Medical Electronics, Vol.BME-8,pp.59-67, January 1961.

35. R.W. Jones and C.C. Li, "Integral Pulse frequency modulated control systems," Paper 417, 2nd Congress IFAC, Basle, August 1963.

36. R.L. Farrankopf, A.E. Sabroff and P.C.Wheeler, "Integral Pulse Frequency on-off control," AIAA 63-328, Aug. 1963.

37. T. Pavlidis, "Analysis and synthesis of PFM feedback systems," Ph.D. Thesis, University of California, Berkeley, 1964.

38. T. Pavlidis, "Equilibrium, controllability and stability of NPFM control systems", Report No. 64-8, April 1964 Electronics Research Laboratory, Univ. of California, Berkeley, California.

39. J.P.C. Clark, "An analysis of pulse-frequency modulated control systems", Ph.D. dissertation, University of Washington, Seattle, 1965.

40. T. Pavlidis and E.I.Jury, "Analysis of a new class of pulse frequency modulated feedback systems", IEEE Trans. Automatic Control, Vol.AC-10, pp.35-43, Jan.1965.

41. A.U. Meyer, E.I.Jury and T.Pavlidis, Discussion of "Analysis of a new class of pulse frequency modulated feedback systems," April 1965, Vol.AC-10, No.2 IEEE Trans. on Automatic Control.

42. J.G. Blanchard, "On the definition and analysis of pulse-frequency modulated systems," Electronic Research Laboratory, University of California, Berkeley, California Report No.65-14.

43. J.P.C. Clark and E.Noges, "The stability of pulse-frequency modulated closed-loop control systems," 1966, IEEE International Convention Record, Vol.14, Pt.6, pp. 179-185.

44. Yu. N.Chekhovoy, "The dynamics of pulse systems with pulse frequency modulation, Accurate method of analysis," Automatic Control theory works of the Seminar, No.1, Naukova Dumak, Kiev, 1966.

45. T.Pavlidis, "Stability of a class of discontinuous dynamical systems," No.3, June 1966. Information and Control.

46. T.Pavlidis, "Stability of Systems described by differential equations containing impulses," Vol.AC-12, No.1, Feb.1967, IEEE Trans. on Automatic Control.

47. E.I.Jury and J.G. Blanchard, "A nonlinear discrete system equivalance of integral pulse frequency modulated control systems," Vol.AC-12, No.4, August 1967, IEEE Trans. on Automatic Control.

48. R.A. Leuchs, "A circuit design for a pulse frequency modulator," M.S. E.E. Thesis, University of Massachusetts, Amherst, June 1967.

49. F. Bombi and D.Ciscato, "A modified Integral pulse frequency modulator in control systems" (Correspondence paper), Vol.AC-12, No.6, pp.784-784, December 1967.

50. V.M. Kuntsevich and Yu.N. Chekhovoi, "Investigating Stability of pulsed control systems with pulse frequency modulation by the direct method of Lyapunov," No.2 Feb.1967, Automation and Remote Control.

51. O.M.Derzhavin, "Block diagrams for pulse time modulators of type 1," No.4, April 1967, Automation and Remote Control.

52. A. Kh.Gelig, "Stabilization of nonlinear systems with pulse-frequency modulation," No.6, June 1967, Automation and Remote Control.

53. Yu. N. Chekhovoy, "A form of stability of nonlinear pulse systems," No.4, July-Aug. 1967 Engineering Cybernetics.

54. V.I. Dymkov, "Absolute stability of pulse-frequency systems," No.10, October 1967, Automation and Remote Control.

55. V.I. Dymkov, "Periodic States in pulse-frequency systems," No.11, November 1967, Automation and Remote Control.

56. R.C. Dorf and T.Liang, "A time optimal pulse frequency modulated step motor servo system," Proc. Nat'l Electronics Conf. (Chicago, Ill. Oct. 1966).

57. T. Pavlidis, "Optimal Control of pulse-frequency modulated systems," IEEE Trans. Aut. Control, Vol.AC-11, October 1966.

58. S.Onysko and E.Noges, "Optimization of pulse-frequency modulated control systems via modified maximum principle," Vol. AC-13, No.2, April 1968.

Double Modulation

59. C.A. Halijak and J.S. Tripp, "A deterministic study of delta modulation," Oct. 1963, IEEE Trans. on Automatic Control.

60. V.M. Kuntsevich and Yu.N. Chekhovoi, "Stability of Control systems with double (pulse frequency and pulse width) modulation," No.7, July 1967, Automation and Remote

61. L.R. Nardizzi and G.A. Bekey, "Optimization of discrete time systems with combined modulation," IEEE Trans. on Automatic Control, June 1968, Vol.AC-13, No.3.

Statistical Study of Pulse-Frequency Modulation

62. Y.W. Lee, "Statistical theory of communication," pp.240-248, Wiley, New York (1960).

63. S.C. Gupta and E.I. Jury, "Statistical study of pulse-width modulated control systems," 1962, Journal of Franklin Institute, 273, pp.292-321.

64. O.A.Z. Leneman, "Statistical properties of random pulse trains," 1966, IEEE International Convention Record 14, Pt.6, pp.167-172.

65. O.A.Z. Leneman, "Random Sampling of random processes: Optimal linear interpolation," 1966, Journal of Franklin Institute 281, pp. 302-314.

66. O.A.Z. Leneman, "Random sampling of random processes: Impulse processes," No.4, June 1966, Information and Control.

67. F.J. Beutler and O.A.Z. Leneman, "Random sampling of random processes:Stationary point processes," No.4, Aug. 1966, Information and Control.

68. O.A.Z. Leneman and J.B. Lewis, "Random sampling of random processes: Mean square comparasion of various interpolators," IEEE Trans. on Automatic Control, Vol.AC-11, pp.396-403, July 1966.

69. Y.T. Chon, "The effect of pulse frequency modulation on random inputs," M.S. E.E. Thesis, University of Massachusetts, Amherst, Mass. Sept. 1967.

70. Bombi and D. Ciscato, "Noise effects in Integral pulse frequency modulators," Submitted to the IFAC Pulse Symposium, Budapest, Hungary, April 1968.

71. C.E. Hutchinson and Y.T. Chon, "The effect of pulse frequency modulation on noise," Vol.AC-12, No.5, Oct. 1967, (Correspondence paper) IEEE Trans. on Automatic Control.

Reconstruction of Continuous Signals

72. P.A. Franaszek and B. Liu, "Optimum recovery of a continuous signal from periodically nonuniformly spaced samples," Proc. IEEE Symp. on Signal Transmission Processing (May 1965), pp.102-107.

73. "On sampled data systems with cyclically varying rates," Journal of Franklin Institute, Vol.283, Feb. 1967, pp. 102-117.

74. B. Liu and P.A. Franaszek, "Sampling with periodically varying rates," IEEE Trans. on Automatic Control, Feb. 1968, Vol.AC-13, No.1.

75. "Error free recovery of signals from irregularly spaced samples," Society of Industrial and Applied Mathematics Rev.

Some general references on nonlinear sampled data systems

76. R.E. Kalman, "Nonlinear aspects of sampled-data control systems," Proc. Symp. on nonlinear circuit analysis, Brooklyn Polytechnique Institute, N.Y., April 1956,pp.272-313.

77. J. Lasalle and S. Lefschetz, "Stability by Liapunov's direct method with applications," Academic Press New York,1961.

78. S.Lefschetz, "Stability of nonlinear control systems," 1965, Academic Press, N.Y.

79. M.A. Aizerman and F.R. Gantmacher, "Absolute Stability of regulator systems," Holden-day, Inc. San Francisco, 1964.

80. V.A. Yakubovich, "Method of matrix inequalities in the theory of the stability of nonlinear controlled systems III absolute stability with hysteresis nonlinearities," Automation and Remote Control, 26, No.5, 1965.

81. Elias P. Gyftopoulos, "Stability criteria for a class of nonlinear systems," Information and Control 6, pp. 276-296 (1963).

82. A.Kh. Gelig, "Investigation of Stability of Nonlinear Discontinuous Automatic Control Systems with a nonunique equilibrium state," Automation and Remote Control, Vol.25, No.2, Aug. 1964.